Robinson-Schensted-Knuth (RSK) algorithm

non-negative integer matrix $A \xrightarrow{RSK} (P, Q)$

SSYT of same shape

① Start w/ A, create a multiset where (i,j) appears A_{ij} many times. Put this into sequence w_A by sorting lexicographically.

Ex. $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 3 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}$

② Set $P(0) = \emptyset$, $Q(0) = \emptyset$

Assuming $P(t)$ defined, define $P(t+1) = (P(t) \leftarrow (w_A)_{2,t+1})$

Consider position of unique box of $P(t+1) \setminus P(t)$.

Add that box to $Q(t)$ w/ value $(w_A)_{1,t+1}$.

(P,Q) = final result after using all columns of w_A.

P = insertion tableau, Q = recording tableau.

Ex. $w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 3 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}$

t	$P(t)$	$Q(t)$
0	$\emptyset$	$\emptyset$
1	$\begin{array}{|c|}\hline 1 \\ \hline\end{array}$	$\begin{array}{|c|}\hline 1 \\ \hline\end{array}$
2	$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$	$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$
3	$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline\end{array}$	$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline\end{array}$
4	$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$
5	$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$	$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$

6 $\quad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$

7 $\quad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$

8 $\quad \underbrace{\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}}_{= P} \qquad \underbrace{\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 & 2 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}}_{= Q}$

Lemma. P, Q are SSYT.

Pf. P is SSYT since we know Schensted insertion produces SSYT.

Entries in first row of w_A are weakly increasing.

$\Rightarrow$ rows of Q and columns are weakly increasing.

When considering identical recording values, insertion values are added in weakly increasing order. Nesting property for insertion paths

$\Rightarrow$ final boxes are added left to right (strictly). $\square$

Thm. RSK is bijection between sets:

$$\left\{\begin{array}{c}\text{non-negative integer} \\ \text{matrices } A\end{array}\right\} \xleftrightarrow{\text{RSK}} \left\{\begin{array}{c}\text{pairs of SSYT of} \\ \text{same shape}\end{array}\right\}.$$

$$\left\{\begin{array}{l} j \text{ appears in } P \text{ exactly } \sum_i A_{ij} \text{ times} \\ i \text{ appears in } Q \text{ exactly } \sum_j A_{ij} \text{ times}\end{array}\right.$$

Pf. If $A \xrightarrow{\text{RSK}} (P, Q)$, can recover w_A from (P, Q) as follows: last entry of first row of w_A contributes the rightmost occurrence of largest value in Q.

Remove to get Q'. Then undo Schensted insertion on P w/ whatever is in that box, i.e., $P = P' \leftarrow j$)

To show RSK surjective: for any (P,Q), can repeat above to get 2-row matrix w whose top row is weakly increasing.

Need to know: if $w_{1,t} = w_{1,t+1}$ then $w_{2,t} \leq w_{2,t+1}$.

If not, then we have $(T \leftarrow j) \leftarrow k$ where $\underset{\substack{\shortparallel \\ w_{2,t}}}{j} > \underset{\substack{\shortparallel \\ w_{2,t+1}}}{k}$.

We showed that in this situation, final box of $T \leftarrow j$ is to the right of final box of $(T \leftarrow j) \leftarrow k$. This violates our choice of boxes.

$\Rightarrow$ columns of w are lexicographically increasing.

$\Rightarrow \exists A$ s.t. $w = w_A$.

$\square$

Cor (Cauchy identity).

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x) s_\lambda(y)$$

$\leftarrow$ sum over all partitions

Pf. Given non-negative integer matrix A, let

$$m(A) = \prod_{i,j} (x_i y_j)^{A_{ij}}$$

$$\text{LHS} = \prod_{i,j} \sum_{A_{ij} \geq 0} (x_i y_j)^{A_{ij}} = \sum_{A} m(A).$$

$\leftarrow$ sum over all non-negative A

By RSK, $m(A) = x^Q y^P$ where $A \xrightarrow{RSK} (P,Q)$

so $$\sum_A m(A) = \sum_\lambda s_\lambda(x) s_\lambda(y)$$

$\square$

Cor. Schur functions form orthonormal basis for Λ.

Cor. $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$

Pf. $\exists$ constants $a_{\lambda\mu}$ s.t. $h_\mu = \sum_\lambda a_{\lambda\mu} s_\lambda$.

By orthonormality, $a_{\lambda\mu} = \langle h_\mu, s_\lambda \rangle$

By definition, $s_\lambda = \sum_\nu K_{\lambda\nu} m_\nu$, and $K_{\lambda\nu} = \langle s_\lambda, h_\nu \rangle$

By symmetry, $\langle s_\lambda, h_\mu \rangle = \langle h_\mu, s_\lambda \rangle$, so $a_{\lambda\mu} = K_{\lambda\mu}$. $\square$

Thm. If $A \xrightarrow{RSK} (P, Q)$, then $A^T \xrightarrow{RSK} (Q, P)$.

So RSK gives bijection between symmetric non-negative matrices and set of SSYT.

Dual RSK algorithm.

Change row insertion: instead of k bumping leftmost value $> k$, we bump leftmost value $\geq k$.

Thm. Dual RSK gives bijection:

$$\left\{ \begin{array}{c} (0,1)\text{-matrices} \\ A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (P, Q),\ P^T \text{ is SSYT} \\ Q \text{ SSYT} \\ P, Q \text{ same shape} \end{array} \right\}$$

type P = column sums of A

type Q = row sums of A

Cor. (Dual Cauchy identity)

$$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda s_\lambda(x)\, s_{\lambda^T}(y)$$

Lemma. Let ω_y be involution ω on y-variables only.

Then $$\prod_{i,j} (1+x_i y_j) = \omega_y \prod_{i,j} (1-x_i y_j)^{-1}$$

Pf. $$\omega_y \prod_{i,j} (1-x_i y_j)^{-1} = \omega_y \sum_\lambda m_\lambda(x) h_\lambda(y)$$
$$= \sum_\lambda m_\lambda(x) e_\lambda(y)$$
$$\prod_i \prod_j (1+x_i y_j) = \prod_i \sum_n e_n(y) x_i^n = \sum_\lambda e_\lambda(y) m_\lambda(x) \qquad \square$$

Cor. $\omega(s_\lambda) = s_{\lambda^T}$.

Pf. $$\omega_y \sum_\lambda s_\lambda(x) s_\lambda(y) = \omega_y \prod_{i,j} (1-x_i y_j)^{-1}$$
$$= \prod_{i,j} (1+x_i y_j)$$
$$= \sum_\lambda s_\lambda(x) s_{\lambda^T}(y)$$

$s_\lambda(x)$ are linearly ind. $\Rightarrow$ coeff of $s_\lambda(x)$ shows $\omega(s_\lambda) = s_{\lambda^T}$. $\square$