

# Robinson-Schensted-Knuth (RSK) algorithm

non-negative integer matrix  $A \xrightarrow{\text{RSK}} (P, Q)$   
 $\swarrow \searrow$   
 SSYT of same shape

① Start w/  $A$ , create a multiset where  $(i, j)$  appears  $A_{ij}$  many times. Put this into sequence  $w_A$  by sorting lexicographically.

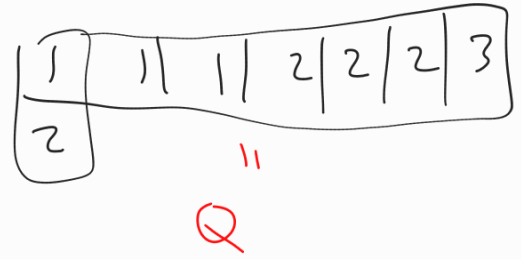
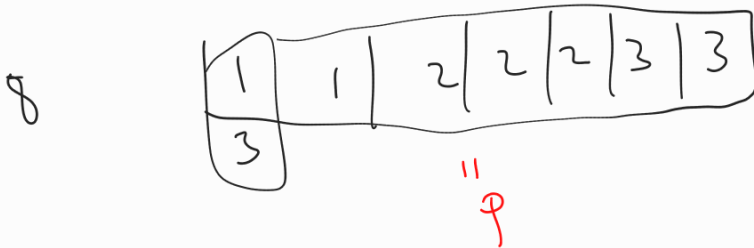
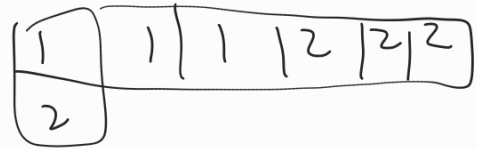
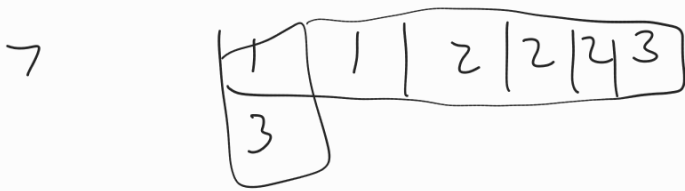
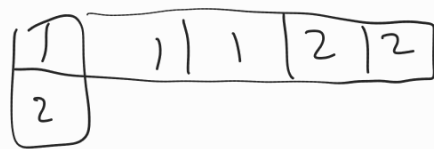
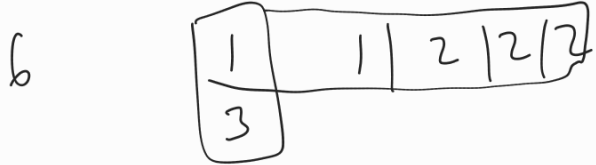
EX.  $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}, w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 3 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}$

② Set  $P(0) = \emptyset, Q(0) = \emptyset$   
 Assuming  $P(t)$  defined, define  $P(t+1) = (P(t) \leftarrow (w_A)_{2,t+1})$   
 Consider position of unique box of  $P(t+1) \setminus P(t)$ .  
 Add that box to  $Q(t)$  w/ value  $(w_A)_{1,t+1}$ .

$(P, Q)$  = final result after using all columns of  $w_A$ .  
 $P$  = insertion tableau,  $Q$  = recording tableau.

EX.  $w_A = \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 3 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}$

$t$	$P(t)$	$Q(t)$
0	$\emptyset$	$\emptyset$
1	$\boxed{1}$	$\boxed{1}$
2	$\boxed{1 \mid 1}$	$\boxed{1 \mid 1}$
3	$\boxed{1 \mid 1 \mid 3}$	$\boxed{1 \mid 1 \mid 1}$
4	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline 3 & & \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 1 \\ \hline 2 & & \end{array}$
5	$\begin{array}{ c c c c } \hline 1 & 1 & 2 & 2 \\ \hline 3 & & & \end{array}$	$\begin{array}{ c c c c } \hline 1 & 1 & 1 & 2 \\ \hline 2 & & & \end{array}$



Lemma. P, Q are SSYT.

Pf. P is SSYT since we know Schensted insertion produces SSYT.

Entries in first row of  $w_P$  are weakly increasing.

$\Rightarrow$  rows of Q and columns are weakly increasing.

When considering identical recording values, insertion values are added in weakly increasing order. Nesting property for insertion paths

$\Rightarrow$  final boxes are added left to right (strictly). □

Thm. RSK is bijection between sets:

$\left. \begin{array}{l} \text{non-negative integer} \\ \text{matrices } A \end{array} \right\} \xleftrightarrow{\text{RSK}} \left. \begin{array}{l} \text{pairs of SSYT of} \\ \text{same shape} \end{array} \right\}$

$\left\{ \begin{array}{l} j \text{ appears in } P \text{ exactly } \sum_i A_{ij} \text{ times} \\ i \text{ appears in } Q \text{ exactly } \sum_j A_{ij} \text{ times} \end{array} \right.$

Pf. If  $A \xrightarrow{\text{RSK}} (P, Q)$ , can recover WA from  $(P, Q)$  as follows: last entry of first row of  $w_A$  contributes the right most occurrence of largest value in Q.

Remove to get  $Q'$ . Then undo Schensted insertion on P w/ whatever is in that box, i.e.,  $P = P' \leftarrow j$

To show RSK surjective: for any  $(P, Q)$ , can repeat above to get 2-row matrix  $w$  whose top row is weakly increasing.

Need to know: if  $w_{1,t} = w_{1,t+1}$  then  $w_{2,t} \leq w_{2,t+1}$ .

If not, then we have  $(T \leftarrow j) \leftarrow k$  where  $j > k$   
" "  
 $w_{2,t}$   $w_{2,t+1}$ .

We showed that in this situation, final box of  $T \leftarrow j$  is to the right of final box of  $(T \leftarrow j) \leftarrow k$ . This violates our choice of boxes.

$\Rightarrow$  columns of  $w$  are lexicographically increasing.

$\Rightarrow \exists A$  s.t.  $w = w_A$ . □

Cor (Cauchy identity).

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y)$$

$\lambda \leftarrow$  sum over all partitions

Pf. Given non-negative integer matrix  $A$ , let

$$m(A) = \prod_{i,j} (x_i y_j)^{A_{ij}}$$

$$\text{LHS} = \prod_{i,j} \sum_{A_{ij} \geq 0} (x_i y_j)^{A_{ij}} = \sum_A m(A)$$

$A \leftarrow$  sum over all non-negative  $A$

By RSK,  $m(A) = x^Q y^P$  where  $A \xrightarrow{\text{RSK}} (P, Q)$

$$\text{so } \sum_A m(A) = \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y)$$
□

Cor. Schur functions form orthonormal basis for  $\underline{\Lambda}$ .

Cor.  $h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$

pf.  $\exists$  constants  $a_{\lambda\mu}$  s.t.  $h_\mu = \sum_\lambda a_{\lambda\mu} s_\lambda$ .

By orthonormality,  $a_{\lambda\mu} = \langle h_\mu, s_\lambda \rangle$

By definition,  $s_\lambda = \sum_\nu K_{\lambda\nu} m_\nu$ , and  $K_{\lambda\nu} = \langle s_\lambda, h_\nu \rangle$

By symmetry,  $\langle s_\lambda, h_\mu \rangle = \langle h_\mu, s_\lambda \rangle$ , so  $a_{\lambda\mu} = K_{\lambda\mu}$ .  $\square$

Thm. If  $A \xrightarrow{RSK} (P, Q)$ , then  $A^T \xrightarrow{RSK} (Q, P)$ .

So RSK gives bijection between symmetric non-negative matrices and set of SSYT.

Dual RSK algorithm.

Change row insertion: instead of  $k$  bumping leftmost value  $> k$ , we bump leftmost value  $\geq k$ .

Thm. Dual RSK gives bijection:

$$\left\{ \begin{array}{l} (0,1)\text{-matrices} \\ A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} (P, Q), \text{ } P^T \text{ is SSYT} \\ Q \text{ SSYT} \\ P, Q \text{ same shape} \end{array} \right\}$$

type  $P$  = column sums of  $A$

type  $Q$  = row sums of  $A$

Cor. (Dual Cauchy identity)

$$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda s_\lambda(x) s_{\lambda^T}(y)$$

Lemma. Let  $\omega_y$  be involution  $\omega$  on  $y$ -variables only.

$$\text{Then } \prod_{i,j} (1+x_i y_j) = \omega_y \prod_{i,j} (1-x_i y_j)^{-1}$$

Pf.  $\omega_y \prod_{i,j} (1-x_i y_j)^{-1} = \omega_y \sum_{\lambda} m_{\lambda}(x) h_{\lambda}(y)$   
 $= \sum_{\lambda} m_{\lambda}(x) e_{\lambda}(y)$

$$\prod_i \prod_j (1+x_i y_j) = \prod_i \sum_n e_n(y) x_i^n = \sum_{\lambda} e_{\lambda}(y) m_{\lambda}(x) \quad \square$$

Cor.  $\omega(s_{\lambda}) = s_{\lambda^T}$ .

Pf.  $\omega_y \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y) = \omega_y \prod_{i,j} (1-x_i y_j)^{-1}$   
 $= \prod_{i,j} (1+x_i y_j)$   
 $= \sum_{\lambda} s_{\lambda}(x) s_{\lambda^T}(y)$

$s_{\lambda}(x)$  are linearly ind.  $\Rightarrow$  coeff of  $s_{\lambda}(x)$  shows  $\omega(s_{\lambda}) = s_{\lambda^T}$ .  $\square$