Schensted insertion

T = SSYT of shape λ, k positive integer

Row / Schensted insertion $T \leftarrow k$ is defined by:

1. Find largest index i s.t. $T_{1,i-1} \leq k$. (if it doesn't exist set $i=1$).
2. Replace $T_{1,i}$ w/ k. If $i = \lambda_1 + 1$, then this means add box to end of first row of λ and put in value k. In this case, result is $T \leftarrow k$, finish. Else, let k' = old value of $T_{1,i}$

   (k <u>bumps</u> k'). Move to next step.
3. Let T' = remove first row of T from T. Compute $T' \leftarrow k'$. Add back new first row of T to $T' \leftarrow k'$.

<u>Insertion path</u> $I(T \leftarrow k)$ is coordinates of bumped boxes.

<u>EX</u>. $T = \begin{array}{cccccc} 1 & 2 & 4 & 5 & 5 & 6 \\ 3 & 3 & 6 & 6 & 8 & \\ 4 & 6 & 8 & & & \\ 7 & & & & & \\ 9 & & & & & \end{array}$, $k = 4$

$$\begin{array}{cccccc} 1 & 2 & 4 & \boxed{5} & 5 & 6 \\ 3 & 3 & 6 & 6 & 8 & \\ 4 & 6 & 8 & & & \\ 7 & & & & & \\ 9 & & & & & \end{array} \leftarrow 4 \qquad \begin{array}{cccccc} 1 & 2 & 4 & 4 & 5 & 6 \\ 3 & 3 & \boxed{6} & 6 & 8 & \\ 4 & 6 & 8 & & & \\ 7 & & & & & \\ 9 & & & & & \end{array} \leftarrow 5$$

$$\begin{array}{cccccc} 1 & 2 & 4 & 4 & 5 & 6 \\ 3 & 3 & 5 & 6 & 8 & \\ 4 & 6 & \boxed{8} & & & \\ 7 & & & & & \\ 9 & & & & & \end{array} \leftarrow 6 \qquad \begin{array}{cccccc} 1 & 2 & 4 & 4 & 5 & 6 \\ 3 & 3 & 5 & 6 & 8 & \\ 4 & 6 & 6 & & & \\ 7 & & & & & \\ 9 & & & & & \end{array} \leftarrow 8$$

$$T \leftarrow k = \begin{array}{cccccc} 1 & 2 & 4 & 4 & 5 & 6 \\ 3 & 3 & 5 & 6 & 8 & \\ 4 & 6 & 8 & & & \\ 7 & 8 & & & & \\ 9 & & & & & \end{array} \qquad I(T \leftarrow k) = \{(1,4),(2,3),(3,3),(4,2)\}.$$

Prop $T \leftarrow k$ is SSYT.

Pf. By construction, rows are weakly increasing.

Claim: If $(i,j), (i+1,j') \in I(T\leftarrow k)$, then $j \geq j'$.

If not, then have

$$\begin{array}{cc} \overset{(i,j)}{\square} & \\ & \overset{(i+1,j')}{\square} \end{array}$$

But $T_{i,j} < T_{i+1,j}$ and $T_{i,j}$ is being inserted into row $i+1$.

This violates construction ~~st~~.

In particular, let $T' = T \leftarrow k$. For each $(i,j) \in I(T\leftarrow k)$,

need to check $T'_{i-1,j} \overset{①}{\leq} T'_{i,j} \overset{②}{<} T'_{i+1,j}$.

Note: ① $T'_{i,j}$ is value bumped from row $i-1$. This was bumped by something smaller, i.e., $T'_{ij} > T'_{i-1,j'}$ where $(i-1,j') \in I(T\leftarrow k)$. By claim, $j' \geq j$, so $T'_{i-1,j'} \geq T'_{i-1,j}$.

② let $(i+1,j'') \in I(T\leftarrow k)$

$T'_{ij} < T_{ij}$. If $j = j''$, then $T_{ij} = T'_{i+1,j}$ ✓

Else, $j'' < j$, so $T'_{i+1,j} = T_{i+1,j}$ ✓ $\square$

Lemma. ("Nested property") Let T = SSYT, $j \leq k$.

Then $I(T\leftarrow j)$ is strictly to left of $I((T\leftarrow j)\leftarrow k)$

i.e., if $(r,s) \in I(T\leftarrow j)$ & $(r,s') \in I((T\leftarrow j)\leftarrow k)$, then $s < s'$.

Furthermore, $\# I(T\leftarrow j) \geq \# I((T\leftarrow j)\leftarrow k)$.

Pf. When inserting k into first row of $T \leftarrow j$, position that k bumps is strictly to the right of position that j bumps.

If j bumps j', k bumps k', then $j' \leq k'$, so rest of first statement follows by recursive nature of row insertion.

For second statement, let $r = \# I(T \leftarrow j)$. Suppose $r \leq \# I((T \leftarrow j) \leftarrow k)$.

Then, consider insertions in row r: for $T \leftarrow j$, value gets added to the end. For $(T \leftarrow j) \leftarrow k$, insertion happens to right of that, i.e., the end of the row, so $r = \# I((T \leftarrow j) \leftarrow k)$. $\square$

Lemma. T = SSYT, $j > k$. If (r,s) is last box of $I(T \leftarrow j)$ and (r', s') last box in $I((T \leftarrow j) \leftarrow k)$, then $s' \leq s$.

Pf. Suppose j is added to end of first row of T.

Since $k < j$, it must bump something in first row of $(T \leftarrow j)$ and so final box must appear weakly to left of end of first row of $(T \leftarrow j)$.

Otherwise, j bumps j' & k bumps k' & $k' \leq j < j'$ and so can reduce to smaller SSYT by removing first row. $\square$

"Reversing" row insertion:

Given SSYT T' and location a value k which is at end of its row, we can undo row insertion: $\exists!$ SSYT T & value i s.t. $T' = (T \leftarrow i)$ and k is last value inserted.