Frobenius characteristic map

CF_n = $\mathbb{Q}$-valued class functions on $\mathfrak{S}_n$

$$(\varphi, \psi)_{\mathfrak{S}_n} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varphi(\sigma) \psi(\sigma)$$

Given $\sigma \in \mathfrak{S}_n$, let $t(\sigma)$ = partition whose parts are cycle lengths of σ (cycle type of σ)

λ partition, $m_i(\lambda)$ = # of times i appears in λ

$$z_\lambda = \prod_i m_i(\lambda)! \, i^{m_i(\lambda)}$$

<u>Lemma</u>. Conjugacy class of λ has size $\frac{n!}{z_\lambda}$.

<u>Pf</u>. Given σ w/ $t(\sigma) = \lambda$, suffices to show centralizer of σ has size z_λ.

Note: each cycle $(i_1, i_2, \ldots, i_k)$ is equal to cyclic shift $(i_2, i_3, \ldots, i_k, i_1)$

There are $\prod_i i^{m_i(\lambda)}$ many ways to permute each cycle cyclically.

Can also permute cycles of the same length.

$\Rightarrow$ z_λ many elements in centralizer $\square$

Let 1_λ be indicator function on conjugacy class of λ

i.e., $1_\lambda(\sigma) = \begin{cases} 1 & \text{if } t(\sigma) = \lambda \\ 0 & \text{else} \end{cases}$.

<u>Cor</u>. $(1_\lambda, 1_\mu)_{\mathfrak{S}_n} = z_\lambda^{-1} \delta_{\lambda\mu}$

<u>Pf</u>. If $\lambda \neq \mu$, then $(1_\lambda, 1_\mu)_{\mathfrak{S}_n} = 0$

Otherwise, $(1_\lambda, 1_\lambda)_{\mathfrak{S}_n} = \frac{1}{n!} c$ where c = size of conjugacy class.

$= 1/z_\lambda$. $\square$

Given n, m. Think of $\mathfrak{S}_n \times \mathfrak{S}_m$ as subgroup of $\mathfrak{S}_{n+m}$:

$$\mathfrak{S}_n = \{\sigma \in \mathfrak{S}_{n+m} \mid \sigma(i) = i \text{ for } i = n+1, \dots, n+m\}$$

$$\mathfrak{S}_m = \{\sigma \in \mathfrak{S}_{n+m} \mid \sigma(i) = i \text{ for } i = 1, \dots, n\}$$

Define <u>induction product</u>:

$$\circ : CF_n \times CF_m \longrightarrow CF_{n+m}$$

$$\varphi \circ \psi = \mathrm{Ind}_{\mathfrak{S}_n \times \mathfrak{S}_m}^{\mathfrak{S}_{n+m}} (\varphi \boxtimes \psi)$$

This gives $CF := \bigoplus_{n \geq 0} CF_n$ a ring structure.

<u>Frobenius characteristic map</u>:

$$\mathrm{ch} : CF_n \longrightarrow \Lambda_{\mathbb{Q}, n}$$

$$\mathrm{ch}(\varphi) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varphi(\sigma)\, p_{t(\sigma)}$$

$$= \sum_{\lambda \in \mathrm{Par}(n)} z_\lambda^{-1} \varphi(\lambda)\, p_\lambda$$

Combine them to get $\mathrm{ch} : CF \longrightarrow \Lambda_{\mathbb{Q}}$.

<u>Prop</u>. ch is an isometry: $(\varphi, \psi)_{\mathfrak{S}_n} = \langle \mathrm{ch}(\varphi), \mathrm{ch}(\psi) \rangle$ for all $\varphi, \psi \in CF_n$.

<u>Pf</u>. $\langle \mathrm{ch}(\varphi), \mathrm{ch}(\psi) \rangle = \langle \sum_\lambda z_\lambda^{-1} \varphi(\lambda) p_\lambda, \sum_\mu z_\mu^{-1} \psi(\mu) p_\mu \rangle$

$$= \sum_\lambda z_\lambda^{-1} \varphi(\lambda) \psi(\lambda) = (\varphi, \psi)_{\mathfrak{S}_n} \qquad \square$$

Prop. ch is a ring isomorphism: $ch(\varphi \circ \psi) = ch(\varphi)\, ch(\psi)$.

Pf. Given λ, μ let $\lambda \cup \mu$ be partition obtained by taking union of all parts in order.

Claim: $1_\lambda \circ 1_\mu = \frac{z_{\lambda\cup\mu}}{z_\lambda z_\mu} 1_{\lambda \cup \mu}$

Let ν be partition of $n+m$. Then:

$$\left(\mathrm{Ind}_{G_n \times G_m}^{G_{n+m}} (1_\lambda \boxtimes 1_\mu), 1_\nu \right)_{G_{n+m}} = \left(1_\lambda \boxtimes 1_\mu, \mathrm{Res}_{G_n \times G_m}^{G_{n+m}} 1_\nu\right)_{G_n \times G_m}$$

$$= \frac{\delta_{\lambda\cup\mu, \nu}}{z_\lambda z_\mu}$$

Write $1_\lambda \circ 1_\mu = \sum_\nu c_\nu 1_\nu$. Then

$(1_\lambda \circ 1_\mu, 1_\nu)_{G_{n+m}} = c_\nu / z_\nu$ $\Rightarrow c_\nu = \frac{z_\nu \delta_{\lambda\cup\mu,\nu}}{z_\lambda z_\mu}$

$\Rightarrow 1_\lambda \circ 1_\mu = \frac{z_{\lambda\cup\mu}}{z_\lambda z_\mu} 1_{\lambda\cup\mu}$. ✓

$ch(1_\lambda) = p_\lambda / z_\lambda$, so

$ch(1_\lambda \circ 1_\mu) = \frac{z_{\lambda\cup\mu}}{z_\lambda z_\mu} ch(1_{\lambda\cup\mu}) = \frac{p_{\lambda\cup\mu}}{z_\lambda z_\mu}$

$ch(1_\lambda)\, ch(1_\mu) = \frac{p_\lambda p_\mu}{z_\lambda z_\mu}$

so $ch(1_\lambda \circ 1_\mu) = ch(1_\lambda)\, ch(1_\mu)$

In general, if $\varphi = \sum_\lambda \varphi(\lambda) 1_\lambda$, $\psi = \sum_\mu \psi(\mu) 1_\mu$,

then $ch(\varphi \circ \psi) = \sum_{\lambda,\mu} \varphi(\lambda)\psi(\mu)\, ch(1_\lambda \circ 1_\mu) = \sum_{\lambda,\mu} \varphi(\lambda)\psi(\mu)\, ch(1_\lambda)\, ch(1_\mu)$
$= ch(\varphi)\, ch(\psi)$.

Finally, 1_λ map to a basis for $\Lambda_{\mathbb{Q}}$, so ch is an isomorphism. $\square$

Let $CF_n' \subset CF_n$ be abelian subgroup spanned by characters of representations. (CF_n' is free abelian group w/ basis characters of irreducible representations = Specht modules)

<u>Note:</u> $CF' = \bigoplus_{n \geq 0} CF_n'$ is subring of CF under $\circ$.

Let $1_{\mathfrak{S}_n}$ be trivial character: $1_{\mathfrak{S}_n}(\sigma) = 1$ for all σ.

For partition $\alpha = \underbrace{(\alpha_1, \alpha_2 \ldots \alpha_k)}_{|\alpha| = n}$, define

$$\eta^\alpha = 1_{\mathfrak{S}_{\alpha_1}} \circ \cdots \circ 1_{\mathfrak{S}_{\alpha_k}}.$$

Recall: M^α = permutation representation on α-tabloids
S^α = Specht module

<u>Prop.</u> η^α = character of M^α. So $\eta^\alpha \in CF'$
Furthermore, η^α form basis for CF'.

<u>Pf.</u> From before: $M^\alpha \cong \operatorname{Ind}_{\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_k}}^{\mathfrak{S}_{\alpha_n}}(\text{trivial})$

Also: if $\exists$ nonzero $\mathfrak{S}_n$-equivariant map $S^\lambda \to M^\mu$, then $\lambda \trianglerighteq \mu$
and S^λ appears once in decomposition of M^λ into irreducible reps.

$\Rightarrow$ if $\eta^\alpha = \sum_\beta a_{\alpha\beta}$ (character of S^β)

then $a_{\alpha\beta} \neq 0 \Rightarrow \beta \trianglerighteq \alpha$
and $a_{\alpha\alpha} = 1$

$\Rightarrow$ $(a_{\alpha\beta})$ invertible over $\mathbb{Z}$
$\Rightarrow$ $\{\eta^\alpha \mid |\alpha| = d\}$ basis for CF_d'. $\square$

Prop. $ch(\eta^\alpha) = h_\alpha$. In particular, $ch(CF') = \Lambda$.

Pf. $ch(1_{S_n}) = \sum_\lambda z_\lambda^{-1} p_\lambda = h_n$

Since ch is ring homomorphism, $ch(\eta^\alpha) = h_\alpha$. $\square$

Prop Suppose $\varphi_1, \dots \varphi_{p(n)} \in CF'_n$ are an orthonormal basis.

Then irreducible characters must be $\pm \varphi_1, \dots, \pm \varphi_{p(n)}$.

Pf. Write φ_i as integer linear combinations of irreducible characters.

These coefficients give orthogonal matrix A, i.e. $AA^T = id$.

Each row is unit vector $\Rightarrow$ $\pm$ standard basis vector.

$\Rightarrow$ A is permutation matrix up to signs. $\square$

Goal: Find orthonormal basis for Λ w.r.t. $\langle , \rangle$.