Math 202B, Winter 2022
Homework 3
Due: February 14 11:59PM via Gradescope

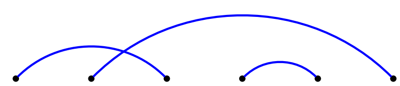
Please do not search for solutions. I would rather help you directly (via office hours or Discord) so that I can calibrate explanations in the notes and lecture. You are free to work with other students, but solutions must be written in your own words. Please cite any sources (beyond the course materials) that you use or any people you collaborated with.

This covers the material up to lecture 15.

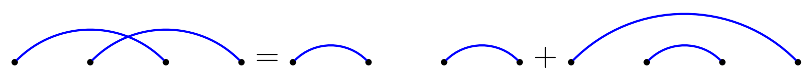
(1) Find a formula (and prove it) for the number of standard Young tableaux for the following families of partitions:
 (a) $(n, 1^k)$ for $n \geq 1$
 (b) $(n, 2)$ for $n \geq 2$

(2) For this problem, work over $\mathbf{C}$.
 (a) Show that $\mathbf{M}^{(n-2,2)} \cong \mathbf{S}^{(n-2,2)} \oplus \mathbf{S}^{(n-1,1)} \oplus \mathbf{S}^{(n)}$ over a field of characteristic 0.
 (b) Using that $\mathbf{M}^{(n-2,2)}$ is isomorphic to a subrepresentation of the degree 2 polynomials in $x_1, \dots, x_n$, identify the 3 subspaces above with spaces of degree 2 polynomials under this isomorphism.

(3) A perfect matching on the set $[2n] = \{1, \dots, 2n\}$ is a decomposition into disjoint 2 element subsets. We can visualize them listing the numbers $1, \dots, 2n$ from left to right and drawing arcs to join the two elements in the same subset. For example:



represents $\{\{1,3\},\{2,6\},\{4,5\}\}$. Let $\mathbf{k}$ be any field and let V_n be the vector space with basis given by all perfect matchings modulo the following relations: for each $a < b < c < d$ if a perfect matching has arcs $\{a, c\}$ and $\{b, d\}$ then it is equal to the sum of the perfect matching with arcs $\{a, b\}$, $\{c, d\}$ and the perfect matching with arcs $\{a, d\}$, $\{b, c\}$ (and all other arcs remain the same in both cases). Visually, this looks like (here we show 4 increasing values $a < b < c < d$ and ignore what happens for the remaining arcs which stay the same in all 3 diagrams):



There is a natural $\mathfrak{S}_{2n}$-action on V_n; show that it is isomorphic to the Specht module $\mathbf{S}^{(n,n)}$ and identify what the standard polytabloids map to under your isomorphism.

(4) (a) Let q be a variable. Show that

$$h_d(1, q, \dots, q^n) = h_{d-1}(1, q, \dots, q^n) + q^d h_d(1, q, \dots, q^{n-1}).$$

 (b) Define $[n]_q = \dfrac{1-q^n}{1-q}$, $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q![n-k]_q!}$.

 Show that $h_d(1, q, \dots, q^n) = \begin{bmatrix} d+n \\ d \end{bmatrix}_q$.

 (c) Show that

$$e_d(1, q, \dots, q^n) = q^{\binom{d}{2}} \begin{bmatrix} n+1 \\ d \end{bmatrix}_q.$$

(5) Prove the following identities

$$nh_n = \sum_{i=1}^{n} p_i h_{n-i}, \qquad ne_n = \sum_{i=1}^{n} (-1)^{i-1} p_i e_{n-i}.$$

Hint: use Lemma 3.6.1 and $\frac{d}{dt}\log F(t) = F'(t)/F(t)$ for any power series F with nonzero constant term.

(6) Consider elements of the tensor product $\Lambda \otimes \Lambda$ as functions in two sets of variables $x_1, x_2, \dots$ and $y_1, y_2, \dots$ which are symmetric under permuting the x_i and also symmetric under permuting the y_j (but not necessarily under permuting all of them together). Define

$$\Delta\colon \Lambda \to \Lambda \otimes \Lambda$$

as follows: for $f \in \Lambda$, we substitute in $x_1, x_2, \dots, y_1, y_2, \dots$ into the variables (in any order since f is symmetric!), which is an element of $\Lambda \otimes \Lambda$. This is a ring homomorphism (you don't need to prove this).

So for example,

$$\Delta(m_1) = x_1 + x_2 + \cdots + y_1 + y_2 + \cdots = m_1(x) + m_1(y).$$

(a) Let d be a non-negative integer. Prove the following identity:

$$\Delta(h_d) = \sum_{i=0}^{d} h_i(x) h_{d-i}(y).$$

(b) Let d be a non-negative integer. Find a formula for $\Delta(e_d)$.

(c) Let d be a positive integer. Find a formula for $\Delta(p_d)$.

(d) Let λ be a partition. Find a formula for $\Delta(m_\lambda)$.

1. Extra problems (don't turn in)

(7) Find and prove a formula for the number of standard Young tableaux of shape (n, n).

(8) Write the polytabloid e_t for $t = \begin{matrix} 4 & 1 \\ 3 & 2 \end{matrix}$ as a linear combination of standard polytabloids.

(9) This exercise is to familiarize you with formal power series rings. Let R be a commutative ring with multiplicative identity 1. Let $R[\![t]\!]$ denote the set of formal linear combinations $\sum_{n\geq 0} r_n t^n = r_0 + r_1 t + \cdots$. These are called formal power series. Two are equal if and only if the coefficients are all pairwise equal. We define $[t^n] \sum_{n\geq 0} r_n t^n = r_n$.

(a) Show that $R[\![t]\!]$ is a commutative ring with the operations

$$\sum_{n\geq 0} r_n t^n + \sum_{n\geq 0} s_n t^n = \sum_{n\geq 0} (r_n + s_n) t^n$$

$$\Big(\sum_{n\geq 0} r_n t^n\Big)\Big(\sum_{n\geq 0} s_n t^n\Big) = \sum_{n\geq 0} \Big(\sum_{i=0}^{n} r_i s_{n-i}\Big) t^n.$$

(b) Show that $\sum_{n\geq 0} r_n t^n$ has a multiplicative inverse if and only if r_0 has a multiplicative inverse in R. (If r_0 is invertible, the identity $(\sum_{n\geq 0} r_n t^n)(\sum_{n\geq 0} s_n t^n) = 1$ gives a system of equations for the s_n which can be solved by induction.)

(c) If $F_1(t), F_2(t), \dots$ are formal power series, we say that the sequence converges to $F(t)$ if for each n, we have $[t^n]F_i(t) = [t^n]F(t)$ for all but finitely many i. In that case we write

$$\lim_{i \to \infty} F_i(t) = F(t).$$

This allows us to define infinite sums as limits of partial sums, i.e.,

$$\sum_{i=1}^{\infty} F_i(t) = \lim_{j \to \infty} \sum_{i=1}^{j} F_i(t)$$

when it exists and also infinite products

$$\prod_{i=1}^{\infty} F_i(t) = \lim_{j \to \infty} \prod_{i=1}^{j} F_i(t).$$

Show that an infinite sum, if it exists, can be rearranged arbitrarily and give the same value. Show the same for infinite products.
If R also has a notion of convergence, then we can take that into account and define the limit to be

$$F(t) = \sum_{n \geq 0} \left(\lim_{i \to \infty} [t^n]F_i(t) \right) t^n.$$

This is relevant when R is the ring of bounded degree power series where convergence is defined by asking that the coefficient of each term is constant for $i \gg 0$. Show that the rearrangement properties continue to hold for this example.

(d) Let $F(t) = \sum_{n \geq 0} f_n t^n$ and $G(t)$ be formal power series such that $G(t)$ has no constant term. Define the composition to be

$$F(G(t)) = \sum_{n \geq 0} f_n G(t)^n.$$

Explain why this is well-defined. We define $\exp(G(t)) = F(G(t))$ where $F(t) = \sum_{n \geq 0} t^n/n!$. Show that

$$\exp(\sum_{i \geq 0} F_i(t)) = \prod_{i \geq 0} \exp(F_i(t))$$

assuming that $\sum_{i \geq 0} F_i(t)$ is well-defined.

(e) We define derivatives of formal power series in the obvious way: $F'(t) = \sum_{n \geq 1} n f_n t^{n-1}$. All of the familiar rules hold (chain rule, product rule, etc.) Many of the familiar Taylor series from calculus have analogues in the world of formal power series, and for the most part they behave exactly like they do in calculus (e.g., log is the compositional inverse of exp).