

Please do not search for solutions. I would rather help you directly (via office hours or Discord) so that I can calibrate explanations in the notes and lecture. You are free to work with other students, but solutions must be written in your own words. Please cite any sources (beyond the course materials) that you use or any people you collaborated with.

This covers the material in lectures 1–4.

- (1) Let G be a finite group and let V, W be finite-dimensional G -representations. Define

$$\Phi: V^* \otimes W \rightarrow \text{Hom}(V, W)$$
$$\Phi\left(\sum_i f_i \otimes w_i\right) = F, \quad \text{where } F(v) = \sum_i f_i(v)w_i.$$

Here $f_i \in V^*$ and $w_i \in W$ are arbitrary. Show that Φ is well-defined and is a G -equivariant isomorphism.

- (2) Let G be a finite abelian group and let V be an irreducible representation over an algebraically closed field (of arbitrary characteristic). Use Schur's lemma to prove that $\dim V = 1$.
- (3) Let G be a group. Define $[G, G]$ to be the subgroup of G generated by elements of the form $xyx^{-1}y^{-1}$ where $x, y \in G$.
- (a) Show that $[G, G]$ is a normal subgroup and that $G/[G, G]$ is abelian.
- (b) Show that $[G, G]$ is in the kernel of any representation $\rho: G \rightarrow \mathbf{GL}(V)$ where $\dim(V) = 1$ and deduce that there is a bijection between the 1-dimensional representations of G and of $G/[G, G]$.
- (4) Let X be a set with G -action and let $V = \mathbf{C}[X]$ be the permutation representation. Let χ_1 be the character of the trivial representation.
- (a) Show that (χ_V, χ_1) is the number of orbits of G acting on X .
- (b) For the rest of the problem, assume that $|X| \geq 2$ and that the action is transitive. The line spanned by $\sum_{x \in X} e_x$ is a subrepresentation, let U be a subrepresentation of $\mathbf{C}[X]$ which is a complement of it. Show that $(\chi_U, \chi_1) = 0$.
- (c) Define an action of G on $X \times X$ by $g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$. Show that $\chi_{\mathbf{C}[X \times X]} = \chi_V^2$.
- (d) Show that U is irreducible if and only if G has exactly 2 orbits on $X \times X$.
- (5) Let \mathbf{F} be a field, let $G = \mathbf{GL}_2(\mathbf{F})$ be the group of invertible 2×2 matrices with entries in \mathbf{F} , and let X be the set of lines, i.e., 1-dimensional subspaces in \mathbf{F}^2 . This has a natural action of G by $g \cdot \ell = \{g \cdot x \mid x \in \ell\}$. Show that $X \times X$ has exactly 2 orbits. When \mathbf{F} is finite, the representation U from above is called the **Steinberg representation** of G .
- (6) Let $n > 1$ and let \mathbf{k} be a field. Prove that $\{(x_1, \dots, x_n) \in \mathbf{k}^n \mid x_1 + \dots + x_n = 0\}$ is an irreducible representation of the symmetric group \mathfrak{S}_n when \mathbf{k} has characteristic 0. Show that this remains true if \mathbf{k} has characteristic $p > 0$ and p does not divide n . What happens when p divides n ?

Extra problems (don't turn in).

- (7) Pick bases for representations V and W . Find bases for $V \oplus W$, V^* , and $V \otimes W$, and describe what the matrices for each $g \in G$ look like, with the goal of finding simple descriptions in terms of the original matrices.
- (8) Let G be a finite group. If the characteristic of \mathbf{k} is positive and divides $|G|$, show that the line spanned by $\sum_{g \in G} e_g$ in $\mathbf{k}[G]$ is a subrepresentation which is not a direct summand as a representation (i.e., there is no complement which is also a subrepresentation).