$a_n = c_1 a_{n-2} + c_2 a_{n-2}$ $\qquad n \geq 2.$

Solve for a_n:

1. Factor $t^2 - c_1 t - c_2 = (t - r_1)(t - r_2)$
2. (a) If $r_1 \neq r_2$, $\exists\ \alpha_1, \alpha_2$ s.t.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

   (b) If $r_1 = r_2$, $\exists\ \alpha_1, \alpha_2$ s.t.

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n$$

3. Solve for α_1, α_2 using $n=0$, $n=1$.

General order d recurrence relations:

$$a_n = c_1 a_{n-1} + \dots + c_d a_{n-d}$$

char. poly. $t^n - c_1 t^{n-1} - c_2 t^{n-2} - \dots - c_d$

factor as $(t - r_1)(t - r_2) \cdots (t - r_d)$

If all $r_1, \dots, r_d$ different, $\exists\ \alpha_1, \dots, \alpha_d$ s.t.

$$a_n = \alpha_1 r_1^n + \dots + \alpha_d r_d^n$$

Ex. $d = 5$ (repeated roots)

$r_1 = r_2 = r_3$, $r_4 = r_5$ $\qquad r_1 \neq r_4$

$\exists\ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ s.t.

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n + \alpha_3 n^2 r_1^n + \alpha_4 r_4^n + \alpha_5 n r_4^n$$

Non-homogeneous?

Ex. $a_n = a_{n-1} + a_{n-2} + 2$ $\qquad$ for $n \geq 2$

Approach 1: $A(x) = \sum_{n \geq 0} a_n x^n$

$$A(x) = a_0 + a_1 x + \sum_{n \geq 2} a_n x^n$$
$$= a_0 + a_1 x + \sum_{n \geq 2} (a_{n-1} + a_{n-2} + 2) x^n$$

$$= a_0 + a_1 x + x(A(x) - a_0) + x^2 A(x) + 2\sum_{n\geq 2} x^n$$

$$= a_0 + a_1 x + \underline{x(A(x) - a_0) + x^2 A(x)} + 2x^2 \sum_{n\geq 0} x^n$$

$$\frac{2x^2}{1-x}$$

Approach 2 (suppose $n \geq 3$).

$$\begin{array}{rl} a_n = & a_{n-1} + a_{n-2} + 2 \\ -(a_{n-1} = & a_{n-2} + a_{n-3} + 2) \\ \hline a_n - a_{n-1} = & a_{n-1} - a_{n-3} \end{array}$$

Initial: a_0, a_1

$\underline{a_2 = a_1 + a_0 + 2}$

$$a_n = 2a_{n-1} - a_{n-3} \quad (n \geq 3)$$

Other examples:

$$\begin{array}{rl} a_n = & a_{n-1} + a_{n-2} + 2^n \\ -2(a_{n-1} = & a_{n-2} + a_{n-3} + 2^{n-1}) \\ \hline a_n - 2a_{n-1} = & a_{n-1} - a_{n-2} - 2a_{n-3} \end{array}$$

$$\begin{array}{rl} a_n = & a_{n-1} + a_{n-2} + n \\ -(a_{n-1} = & a_{n-2} + a_{n-3} + n - 1) \\ \hline a_n = & 2a_{n-1} - a_{n-3} + 1 \\ a_{n-1} = & 2a_{n-2} - a_{n-4} + 1 \\ \hline a_n - a_{n-1} = & 2a_{n-1} - 2a_{n-2} - a_{n-3} - a_{n-4} \end{array}$$

$(n \geq 4)$

Combinatorial Interpretations of operations on OGFs

Idea: Interpret a_n = number of "structures" on the set of $[n]$ of some type

Ex. $a_n = n!$ structure = orderings of elements of $[n]$

$$\sum_{n\geq 0} n!\, x^n = \text{OGF for orderings}$$

$b_n = 2^n$ structure = Choice of subset of $[n]$

Addition of OGF $\longleftrightarrow$ "OR" operation on structures.

Ex. ① $a_n + b_n = n! + 2^n$ structure = ordering $[n]$ OR picking subset of $[n]$

② $a_n + a_n = 2n!$ structure = person 1 orders $[n]$ OR person 2 orders $[n]$

Products of OGF $\longleftrightarrow$ "Concatenation" of structures.

$$A(x)B(x) = \sum_{n \geq 0} c_n x^n$$

$$c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

structure = ways to break (n) into 2 consecutive pieces $[i] \cup \{i+1, \ldots, n\}$
+ putting structure A on $[i]$
+ putting structure B on $\{i+1, \ldots, n\}$.

Ex. A class has n days.
Split schedule into 2 pieces: lecture, lab
1 guest lecture in first part
2 guest labs in second part.
How many ways to design this course?

Structure 1: picking one lecture to be guest lecture

$a_n = n$, $A(x) = \sum_{n \geq 0} n x^n$

Structure 2: picking 2 labs to guest labs

$b_n = \binom{n}{2}$, $B(x) = \sum_{n \geq 0} \binom{n}{2} x^n$

Answer = $[x^n] A(x) B(x)$.

$$A(x) = \sum_{n\geq 0} n x^n = \frac{x}{(1-x)^2}$$

$$B(x) = \sum_{n\geq 0} \frac{n(n-1)}{2} x^n = \frac{x^2}{2} D^2\left(\frac{1}{1-x}\right) = \frac{x^2}{(1-x)^3}$$

$$A(x)B(x) = \frac{x^3}{(1-x)^5} = x^3 (1-x)^{-5}$$

$$\binom{-d}{n} = (-1)^n \binom{d+n-1}{n}$$

$$= x^3 \sum_{n\geq 0} \binom{-5}{n}(-x)^n = \sum_{n\geq 0} \binom{n+4}{n} x^{n+3}$$

$$[x^n]\, A(x)B(x) = \binom{(n-3)+4}{4} = \binom{n+1}{4}$$

$$D^2 \frac{1}{1-x} = D^2 \sum_{n\geq 0} x^n = \sum_{n\geq 0} n(n-1) x^{n-2}$$

Partition generating functions

$p_{\leq k}(n) =$ # partitions of n using $\leq k$ parts

$=$ # partitions of n s.t. all parts are $\leq k$

$= \# \left\{ (\lambda_1, \lambda_2, \ldots) \;\middle|\; \begin{array}{l} k \geq \lambda_1 \geq \lambda_2 \geq \ldots \\ \lambda_1 + \lambda_2 + \ldots = n \end{array} \right\}$.

Fix k, let n vary. $\sum_{n\geq 0} p_{\leq k}(n) x^n = ?$ $\quad p_{\leq k}(0) = 1$

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k=1 $\quad p_{\leq 1}(n) = 1$ for all n

$$\sum_{n\geq 0} p_{\leq 1}(n)\, x^n = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

$\underline{k=2}$. Write n as a sum of 1's and 2's.

Consider product

$$(1+x+x^2+x^3+\cdots)(1+x^2+(x^2)^2+(x^2)^3+\cdots)$$

Each term is of the form $x^a \cdot (x^2)^b = x^{a+2b}$

coeff of x^n in this product $= \#\{(a,b) \mid a+2b=n\}$
$= p_{\leq 2}(n)$

$$(a,b) \longleftrightarrow \underbrace{2+\cdots+2}_{b}+\underbrace{1+1+\cdots+1}_{a} \qquad (\underbrace{2,2,\ldots 2}_{b},\underbrace{1,\ldots 1}_{a})$$

$$\sum_{n\geq 0} p_{\leq 2}(n)\, x^n = (1+x+x^2+\cdots)(1+x^2+(x^2)^2+\cdots)$$
$$= \frac{1}{1-x}\cdot\frac{1}{1-x^2} = \frac{1}{(1-x)(1-x^2)}$$

$\underline{\text{General } k}$: $\boxed{\text{Claim: } \sum_{n\geq 0} p_{\leq k}(n)\, x^n = \frac{1}{(1-x)(1-x^2)\cdots(1-x^k)}}$

$\underline{\text{RHS}}$: $(1+x+x^2+\cdots)(1+x^2+(x^2)^2+\cdots)\cdots(1+x^k+(x^k)^2+\cdots)$

Each term is of the form $x^{a_1}(x^2)^{a_2}\cdots(x^k)^{a_k}$

coeff of $x^n = \#\{(a_1,\ldots,a_k) \mid a_1+2a_2+3a_3+\cdots+ka_k=n\}$

$\updownarrow$

partitions of n w/ all parts $\leq k$

$(a_1,\ldots a_k) \longleftrightarrow$ partition where i is used a_i times

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How about $\sum_{n\geq 0} p(n)\, x^n$ $\longrightarrow$ $p(n) = \#$ partitions of n.

Idea: let $k \to \infty$.

Thm (Euler) $\sum_{n\geq 0} p(n) x^n = \prod_{i\geq 1} \frac{1}{1-x^i}$

What is infinite product?

Setup. $A_1(x), A_2(x), \ldots$ are FPS w/ constant term 1.

Define $A_1(x) A_2(x) \ldots$ as follows: coeff of x^n is all ways of multiplying out terms to get x^n

i.e., $(1+\ldots)(1+\ldots)(1+\ldots)\ldots$

we have to choose 1 all but finitely many times.

$$\prod_{i\geq 1} (1-x^i)^{-1} = (1+x+x^2+\ldots)(1+x^2+(x^2)^2+\ldots)(1+x^3+(x^3)^2+\ldots)\ldots$$

why valid?

what is coeff of x^n?

I'll never get x^n if I don't choose 1 from

$\frac{1}{1-x^d}$ whenever $d>n$

$$\Rightarrow [x^n] \prod_{i\geq 1} (1-x^i)^{-1} = [x^n] \prod_{i=1}^{n} (1-x^i)^{-1} = [x^n] \sum_{m\geq 0} p_{\leq n}(m) x^m$$

$$= p_{\leq n}(n) = p(n)$$

Another way: $(1+x+x^2+\ldots)(1+(x^2)^2 + \ldots)\ldots$

each term is of the form $x^{a_1}(x^2)^{a_2}(x^3)^{a_3}\ldots$

coeff of $x^n = \#\{(a_1,a_2,a_3,\ldots) \mid a_1+2a_2+3a_3+\ldots = n\}$ all but finitely many a_i are nonzero

$= p(n)$

Let S be any subset of positive integers.

$p_S(n) =$ # partitions of n s.t. all parts belong to S.

__Thm__. $$\sum_{n \geq 0} p_S(n) x^n = \prod_{i \in S} \frac{1}{1-x^i}.$$