General Binomial Thm: m rational,

$$(1+x)^m = \sum_{n\geq 0} \binom{m}{n} x^n \qquad \binom{m}{n} = \frac{m(m-1)\cdots(m-n+1)}{n!}$$

Ex. $m=-1$: $$(1+x)^{-1} = \frac{1}{1+x} = \sum_{n\geq 0} (-x)^n = \sum_{n\geq 0} (-1)^n x^n$$

Binomial Thm: $$(1+x)^{-1} = \sum_{n\geq 0} \binom{-1}{n} x^n$$

$$\binom{-1}{n} = \frac{(-1)(-2)(-3)\cdots(-1-n+1)}{n!} = \frac{(-1)^n n!}{n!} = (-1)^n$$

$m=-d$, d positive integer

$$(1+x)^{-d} = \left(\frac{1}{1+x}\right)^d = \left(\sum_{n\geq 0} (-1)^n x^n\right)^d = \sum_{n\geq 0} (-1)^n \binom{d+n-1}{n} x^n$$

$$\binom{-d}{n} = \frac{(-d)(-d-1)(-d-2)\cdots(-d-n+1)}{n!} = (-1)^n \frac{d(d+1)(d+2)\cdots(d+n-1)}{n!}$$

$$= (-1)^n \frac{(d+n-1)!}{(d-1)!\, n!} = (-1)^n \binom{d+n-1}{n}$$

substitute $x \to -x$: $$\left(\frac{1}{1-x}\right)^d = \sum_{n\geq 0} \binom{d+n-1}{n} x^n$$

$m = 1/2$ $$(1+x)^{1/2} = \sqrt{1+x}$$

$$\binom{1/2}{n} = \frac{(1/2)(-1/2)(-3/2)\cdots(1/2-n+1)}{n!} = \frac{(-1)^{n-1}}{2^n} \cdot \frac{1\cdot 1\cdot 3\cdot 5\cdots(2n-3)}{n!} \quad (*)$$

Define double factorial: $$n!! = \begin{cases} n(n-2)(n-4)\cdots 2 & \text{if } n \text{ even} \\ n(n-2)(n-4)\cdots 3\cdot 1 & \text{if } n \text{ odd} \end{cases}$$

$$n!! = \prod_{i=0}^{\lfloor n/2 \rfloor} (n-2i)$$

$$(*) = \frac{(-1)^{n-1}(2n-3)!!}{2^n\, n!}$$

<u>Meaning:</u> $$\left(\sum_{n\geq 0} \frac{(-1)^{n-1}(2n-3)!!}{2^n\, n!} x^n\right)^2 = 1+x$$

Quadratic formula: suppose $F(x)$ is FPS which satisfies equation

$$A(x)F(x)^2 + B(x)F(x) + C(x) = 0$$

A, B, C are FPS.

Then $F(x)$ is solution to $A(x)t^2 + B(x)t + C(x) = 0$.

Quadratic formula gives two "solutions":

$$\frac{-B(x) \pm (B(x)^2 - 4A(x)C(x))^{1/2}}{2A(x)}$$

Trick: There is a choice of sign so that $F(x)$ is given by this expression.

Ordinary generating functions (OGF)

$a_0, a_1, a_2, \ldots$ sequence of numbers.

OGF of this sequence is FPS $A(x) = \sum_{n\geq 0} a_n x^n$.

Linear recurrence relations (homogeneous)

A sequence $(a_n)_{n\geq 0}$ satisfies linear recurrence relation (of order d) if $\exists$ constants $c_1, \ldots, c_d$ s.t. $c_d \neq 0$ and

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d} \quad \text{for all } n \geq d.$$

Ex. F_n Fibonacci numbers $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

i.e., $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

($c_1 = 1$, $c_2 = 1$, order 2)

this plus initial values $F_0 = 1$, $F_1 = 1$ determines sequence.

Ex. order 1 recurrence: $a_n = c_1 a_{n-1}$ for all $n \geq 1$.

Solve: $a_n = c_1 a_{n-1} = c_1^2 a_{n-2} = c_1^3 a_{n-3} = \cdots = c_1^n a_0$

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order 2: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Define characteristic polynomial to be $t^2 - c_1 t - c_2 = (t - r_1)(t - r_2)$

Let r_1, r_2 be its roots $\dfrac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2}$

Thm If $r_1 \neq r_2$, Then $\exists$ constants α_1, α_2 s.t.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \text{for all } n \geq 0.$$

To solve for α_1, α_2, plug in $n = 0$, $n = 1$:

$$a_0 = \alpha_1 + \alpha_2$$

$$a_1 = \alpha_1 r_1 + \alpha_2 r_2.$$

Ex. $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$

char. poly. $t^2 - t - 1$ roots are $\dfrac{1 \pm \sqrt{5}}{2}$

$r_1 = \dfrac{1 + \sqrt{5}}{2}$, $r_2 = \dfrac{1 - \sqrt{5}}{2}$. $r_2 - r_1 = \dfrac{1 - \sqrt{5} - 1 - \sqrt{5}}{2} = -\sqrt{5}$

$$\boxed{F_n = \alpha_1 r_1^n + \alpha_2 r_2^n \quad \text{for all } n \geq 0}$$

$$\begin{cases} 1 = F_0 = \alpha_1 + \alpha_2 & \Rightarrow \quad \alpha_1 = 1 - \alpha_2 \\ 1 = F_1 = \alpha_1 r_1 + \alpha_2 r_2 \end{cases}$$

$$\rightarrow \quad 1 = (1-\alpha_2) r_1 + \alpha_2 r_2 = r_1 + \alpha_2 (r_2 - r_1)$$

$$= \frac{1+\sqrt{5}}{2} - \alpha_2 \sqrt{5}$$

$$\alpha_2 \sqrt{5} = \frac{-1+\sqrt{5}}{2} \Rightarrow \alpha_2 = \frac{-1+\sqrt{5}}{2\sqrt{5}}$$

$$\alpha_1 = 1 - \alpha_2 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$F_n = \frac{1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{-1+\sqrt{5}}{2\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}$$

Pf of Thm. Set $A(x) = \sum_{n \geq 0} a_n x^n$.

$$A(x) = a_0 + a_1 x + \sum_{n \geq 2} a_n x^n$$

$$= a_0 + a_1 x + \sum_{n \geq 2} (c_1 a_{n-1} + c_2 a_{n-2}) x^n$$

$$= a_0 + a_1 x + c_1 \sum_{n \geq 2} a_{n-1} x^n + c_2 \sum_{n \geq 2} a_{n-2} x^n$$

$$= a_0 + a_1 x + c_1 x \sum_{n \geq 2} a_{n-1} x^{n-1} + c_2 x^2 \sum_{n \geq 2} a_{n-2} x^{n-2}$$

$m = n-1 \qquad p = n-2$

$$= a_0 + a_1 x + c_1 x \underbrace{\sum_{m \geq 1} a_m x^m}_{A(x) - a_0} + c_2 x^2 \underbrace{\sum_{p \geq 0} a_p x^p}_{A(x)}$$

$$A(x) = a_0 + a_1 x + c_1 x (A(x) - a_0) + c_2 x^2 A(x)$$

$$A(x) - c_1 x A(x) - c_2 x^2 A(x) = a_0 + a_1 x - c_1 a_0 x$$

$$A(x)(1 - c_1 x - c_2 x^2) = a_0 + a_1 x - c_1 a_0 x$$

$$A(x) = \frac{a_0 + a_1 x - c_1 a_0 x}{1 - c_1 x - c_2 x^2} \qquad (*)$$

Factor denominator

$$t^2 - c_1 t - c_2 = (t - r_1)(t - r_2)$$

substitute $t \to 1/x$, multiply by x^2

$$x^2\left(\frac{1}{x^2} - c_1 \frac{1}{x} - c_2\right) = x^2\left(\frac{1}{x} - r_1\right)\left(\frac{1}{x} - r_2\right)$$

$$1 - c_1 x - c_2 x^2 = (1 - r_1 x)(1 - r_2 x)$$

$$(*) \qquad A(x) = \frac{a_0 + (a_1 - c_1 a_0) x}{(1 - r_1 x)(1 - r_2 x)}$$

PARTIAL FRACTION

for some α_1, α_2

$$= \frac{\alpha_1}{1 - r_1 x} + \frac{\alpha_2}{1 - r_2 x} \qquad (\text{b/c } r_1 \neq r_2)$$

equivalent to

$\alpha_1(1 - r_2 x) + \alpha_2(1 - r_1 x) = a_0 + (a_1 - c_1 a_0) x$

$$= \alpha_1 \sum_{n \geq 0} (r_1 x)^n + \alpha_2 \sum_{n \geq 0} (r_2 x)^n$$

Compare coeffs of x^n

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n. \qquad \square$$

Thm. If $r_1 = r_2$, then $\exists$ constants α_1, α_2 s.t.

$$a_n = \alpha_1 r_1^n + \alpha_2 n r_1^n \quad \text{for all } n \geq 0$$

Solve for α_1, α_2: plug in $n=0$, $n=1$

$$a_0 = \alpha_1$$

$$a_1 = \alpha_1 r_1 + \alpha_2 r_1$$

Where is the difference? Partial fraction decomposition.

$$\frac{a_0 + (a_1 - c_1 a_0)x}{(1-r_1x)(1-r_2x)} \underset{\text{if } r_1 = r_2}{=} \frac{\beta_1}{1-r_1x} + \frac{\beta_2}{(1-r_1x)^2}$$

for some constants β_1, β_2

$$= \beta_1 \sum_{n\geq 0} (r_1x)^n + \beta_2 \sum_{n\geq 0} (n+1)(r_1x)^n \quad \square$$