| balls / boxes | f arbitrary | f injective | f surjective |
|---|---|---|---|
| dist / dist | ① $n^k$ | ② $(n)_k$ | ③ $n!\,S(k,n)$ |
| indist / dist | ④ $\binom{n+k-1}{k}$ | ⑤ $\binom{n}{k}$ | ⑥ $\binom{k-1}{n-1}$ |
| dist / indist | ⑦ $\sum_{i=0}^{n} S(k,i)$ | ⑧ $\begin{cases} 1 & \text{if } n \geq k \\ 0 & \text{if } n < k \end{cases}$ | ⑨ $S(k,n)$ |
| indist / indist | ⑩ $p_{\leq n}(k)$ | ⑪ $\begin{cases} 1 & \text{if } n \geq k \\ 0 & \text{if } n < k \end{cases}$ | ⑫ $p_n(k)$ |

$f: \{k \text{ balls}\} \longrightarrow \{n \text{ boxes}\}$ How many f s.t. ...?

① Words of length k in alphabet of boxes
② Words of length k in alphabet of boxes + no repeated entries
③ Ordered partitions of set of balls into n blocks
④ Weak compositions of k into n parts
⑤ Subsets of size k of set of n boxes
⑥ Compositions of k into n parts
⑦ Partitions of k balls into $\leq n$ blocks
⑨ Partitions of k balls into n blocks
⑩ (Integer) partitions of k into $\leq n$ parts
⑫ (Integer) partitions of k into n parts

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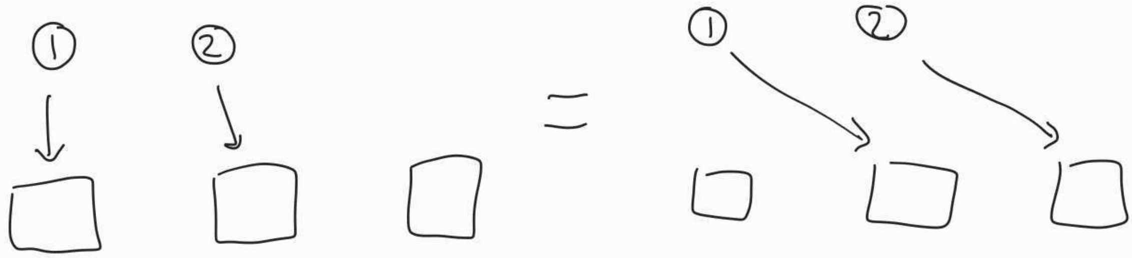
⑧ :

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Binomial Theorem

$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

Thm (Binomial Theorem). For any $n \geq 0$ (integer)

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

Pf 1 (combinatorial).

$$(x+y)^n = \underbrace{(x+y)(x+y) \cdots (x+y)}_{n}$$

Expand it: For each of $n$ $(x+y)$'s, choose either $x$ or $y$.

Multiply these $n$ choices to get a term $x^i y^j$, $(i+j=n)$

Now add together these terms over all possible choices.

Another perspective: Given $x^i y^j$, how many times does it appear?

If $i+j \neq n$, never.

Otherwise, $j = n-i$, get # of times we chose exactly $i$ $x$'s in a sequence of $n$ choices.

Answer is $\binom{n}{i}$. $\square$

Pf 2 (induction). For $n=0$, $(x+y)^0 = 1$

$$\sum_{i=0}^{0} \binom{0}{i} x^i y^{0-i} = \binom{0}{0} x^0 y^0 = 1 \quad \checkmark$$

Induction step (assume $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$)

$$(x+y)^{n+1} = (x+y)(x+y)^n = (x+y)\left(\sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\right)$$

$$= \sum_{i=0}^{n} \binom{n}{i} x^{i+1} y^{n-i} + \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i+1}$$

$j = i+1$

$$= \sum_{j=1}^{n+1} \binom{n}{j-1} x^j y^{n-j+1} + \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i+1}$$

$$= \binom{n}{n} x^{n+1} + \sum_{j=1}^{n} \binom{n}{j-1} x^j y^{n-j+1} + \sum_{i=1}^{n} \binom{n}{i} x^i y^{n-i+1} + \binom{n}{0} y^{n+1}$$

$$= \binom{n}{n} x^{n+1} + \sum_{k=1}^{n} \left( \binom{n}{k-1} + \binom{n}{k} \right) x^k y^{n+1-k} + \binom{n}{0} y^{n+1}$$

Pascal

$$= \binom{n+1}{n+1} x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + \binom{n+1}{0} y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}. \quad \checkmark \qquad \square$$

Cor. $2^n = \sum_{i=0}^{n} \binom{n}{i}$

Pf. Substitute $x = y = 1$ into binomial Thm:

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = \sum_{i=0}^{n} \binom{n}{i}. \qquad \square$$

Cor. For $n > 0$, $0 = \sum_{i=0}^{n} (-1)^i \binom{n}{i}$

Pf. Substitute $x = -1$, $y = 1$ into binomial Thm:

$$0 = (-1+1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i 1^{n-i} \qquad \square$$

Add $\binom{n}{i}$ to both sides for all $i$ odd:

$$\sum_{\substack{0 \le i \le n \\ \text{s.t. } i \text{ odd}}} \binom{n}{i} = \sum_{\substack{0 \le i \le n \\ \text{s.t. } i \text{ even}}} \binom{n}{i}$$

$\Rightarrow$ For $n > 0$, # odd size subsets of $[n]$ = # even size subsets of $[n]$.

Can we describe bijection between set of odd sized subsets & set of even sized subsets?

Two equations: $\#$ even subsets $+$ $\#$ odd subsets $= 2^n$

$\#$ even subsets $=$ $\#$ odd subsets

$\Rightarrow$ $\#$ even subsets $= 2^{n-1} =$ $\#$ odd subsets.

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Start w/ $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$.

Apply $\frac{\partial}{\partial x}$: $n(x+y)^{n-1} = \sum_{i=1}^{n} i \binom{n}{i} x^{i-1} y^{n-i}$

Cor. $n 2^{n-1} = \sum_{i=1}^{n} i \binom{n}{i}$

Pf. Substitute $x=y=1$ into

Interpretation: Sum on RHS $= \#$ ways to choose group from $n$ people + choose leader for that group.

Cor tells us $n2^{n-1}$ ways possible.

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## Multinomial Theorem

$k$ variables $x_1, \ldots, x_k$

Thm. For $n \geq 0$,

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{(a_1, \ldots a_k)} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$$

↳ sum over all weak compositions of $n$ into $k$ parts

Pf. generalize Pf 1 of binomial.

$$(x_1 + \cdots + x_k)^n = \underbrace{(x_1 + \cdots + x_k) \cdots (x_1 + \cdots + x_k)}_{n}$$

Expand: For each $(x_1 + \cdots + x_k)$ choose some $x_i$

Multiply out all choices to get something like $x_1^{a_1} \cdots x_k^{a_k}$

$(a_1 + \cdots + a_k = n)$

Sum together these terms over all possible choices.

Given $a_1, \dots a_k$, how many times does $x_1^{a_1} \cdots x_k^{a_k}$ appear?

If $a_1 + \dots + a_k \neq n$, never.

Else, # ways to arrange $n$ objects w/ $k$ different types s.t. same type considered identical + $a_i$ many of type $i$.

$$= \binom{n}{a_1, a_2, \dots a_k}.$$

$\square$

Freebies:

$$k^n = \sum_{(a_1,\dots,a_k)} \binom{n}{a_1,\dots,a_k}$$

$$0 = \sum_{(a_1,\dots a_k)} (1-k)^{a_1} \binom{n}{a_1,\dots a_k}$$

$$nk^{n-1} = \sum_{(a_1,\dots a_k)} a_1 \binom{n}{a_1,\dots a_k}$$