$$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$$

n\k	1	2	3	4	5
1	1	0	0	0	0
2	1	1	0	0	0
3	1	3	1	0	0
4	1	7	6	1	0
5	1	15	25	10	1

$S(n,k) = 0$ if $k > n$

$B(n) = \#$ partitions of $[n]$, <u>nth Bell number</u>

$$= \sum_{k=0}^{n} S(n,k)$$

<u>Thm</u> $$B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$$

<u>Pf</u>. Separate partitions of $[n+1]$ based on how big the block containing $n+1$ is.

Let j be the size of block containing $n+1$. $(1 \le j \le n+1)$.

These partitions can be constructed as follows:

Pick $j-1$ values from $[n]$ to share block w/ $n+1$.

① Pick $j-1$ values from $[n]$

② Partition remaining values $[n] \setminus \{\text{Choices from ①}\}$

①: $\binom{n}{j-1}$ many ways to choose $j-1$ values.

② $B(n-j+1)$ many ways to choose partition.

$\Rightarrow$ # partitions where block of $n+1$ has size j is

$$\binom{n}{j-1} B(n-j+1).$$

$$\Rightarrow \quad B(n+1) = \sum_{j=1}^{n+1} \binom{n}{j-1} B(n-j+1) \underset{k=j-1}{=} \sum_{k=0}^{n} \binom{n}{k} B(n-k)$$

$$= \sum_{k=0}^{n} \binom{n}{n-k} B(n-k) \underset{i=n-k}{=} \sum_{i=0}^{n} \binom{n}{i} B(i). \qquad \square$$

Integer partitions

n = positive integer. A (integer) partition of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of non-negative integers s.t.

$$\lambda_1 + \cdots + \lambda_k = n \quad \& \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$

$|\lambda| = n$ size of λ

$\ell(\lambda) = \#\{i \mid \lambda_i > 0\}$ length of λ

Convention: Partitions are considered the same if they differ only in 0's

$p(n) = \#$ partitions of n

$p_k(n) = \#$ partitions λ of n s.t. $\ell(\lambda) = k$

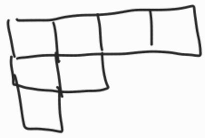
$p_{\leq k}(n) = \#$ partitions λ of n s.t. $\ell(\lambda) \leq k$.

$p(0) = 1$ by convention, let ϕ denote unique partition of 0.

EX.

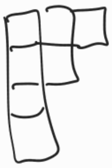
n	partitions of n	$p(n)$
1	(1)	1
2	(2) (1,1)	2
3	(3) (2,1) (1,1,1)	3
4	(4) (3,1) (2,2) (2,1,1) (1,1,1,1)	5
5	(5) (4,1) (3,2) (3,1,1) (2,2,1) (2,1,1,1) (1,1,1,1,1)	7

Visualize partitions using Young diagrams

Ex. $\lambda = (4,2,1)$ $\quad Y(\lambda) =$ 

left-justified boxes.
λ_i boxes in ith row.

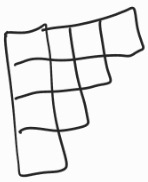
Transpose: λ^T is partition w/ property that
$Y(\lambda^T) =$ flip of $Y(\lambda)$ across diagonal.

Ex. $\lambda = (4,2,1)$, $Y(\lambda^T) =$  , $\lambda^T = (3,2,1,1)$

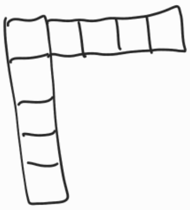
$(\lambda^T)_i = \#\{j \mid \lambda_j \geq i\}$. $\quad (\lambda^T)^T = \lambda$

λ is self-conjugate if $\lambda^T = \lambda$.

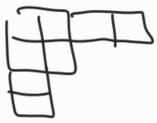
Ex. Self-conjugate partitions: $(4,3,2,1)$, $(5,1,1,1,1)$, $(4,2,1,1)$



"staircase"



(self-conjugate)
"hook"



Thm. # partitions λ of n s.t. $\ell(\lambda) \leq k$ $=$ # partitions μ of n s.t. $\mu_1 \leq k$

$\quad\quad \| $

$\quad p_{\leq k}(n)$

Pf. Bijection between respective sets given by taking transpose.

$$\left\{ \begin{array}{l} \text{partitions } \lambda \text{ of } n \\ \text{s.t. } \ell(\lambda) \leq k \end{array} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{ \begin{array}{l} \text{partitions } \mu \text{ of } n \\ \text{s.t. } \mu_1 \leq k \end{array} \right\},$$

$$f(\lambda) = \lambda^T, \quad g(\mu) = \mu^T.$$

If $\ell(\lambda) \leq k$, then $(\lambda^T)_1 = \ell(\lambda) \leq k$

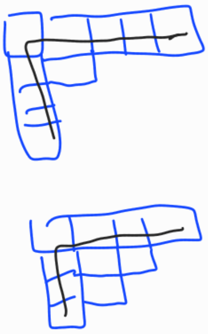
If $\mu_1 \leq k$, then $\ell(\mu^T) = \mu_1 \leq k$. $\square$

Thm. # self-conjugate partitions of n = # partitions of n using distinct odd parts

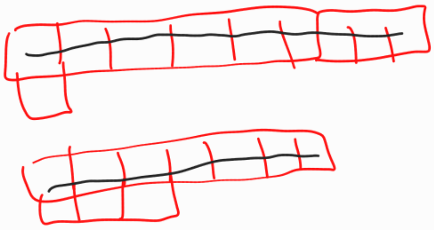
Ex. $n = 10$

$(5,2,1,1,1)$ $\qquad$ $(9,1)$

$(4,3,2,1)$ $\qquad$ $(7,3)$



bend into hooks



Pf. $\left\{ \begin{array}{c} \text{self-conjugate} \\ \text{partitions of } n \end{array} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{ \begin{array}{c} \text{partitions of } n \\ \text{using distinct odd parts} \end{array} \right\}$.

Let λ be self-conjugate.

Remove λ_1 from λ_1 and subtract 1 from remaining entries.

$\rightarrow$ removes $2\lambda_1 - 1$ many boxes from $Y(\lambda)$

We get smaller partition $\mu = (\lambda_2 - 1, \lambda_3 - 1, \ldots)$

Remove μ_1 from μ and subtract 1 from remaining entries.

$\rightarrow$ removed $2\mu_1 - 1$ many boxes from $Y(\mu)$

$\rightsquigarrow$ sequence $f(\lambda) = (2\lambda_1 - 1, 2\mu_1 - 1, \ldots)$ $\rightarrow$ strictly decreasing, i.e. all distinct entries.

Note: $2\mu_1 - 1 = 2(\lambda_2 - 1) - 1 = 2\lambda_2 - 3 \leq 2\lambda_1 - 3 < 2\lambda_1 - 1$.

Constructing g: Start w/ μ, partition of n into odd distinct parts.

Each μ_i is of the form $2x_i - 1$ where $x_i \geq 1$.

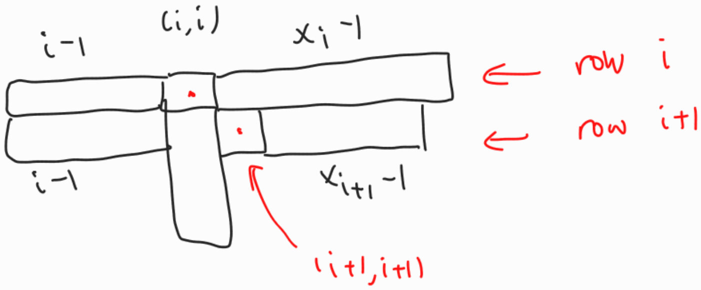
build self-conjugate hook of size μ_i by taking

$$(x_i, \underbrace{1, \ldots 1}_{x_i - 1})$$

Take these hooks and nest them. Call this $g(\mu)$.

(i.e. corner box of ith hook goes in ith row, ith column)

why partition?



$$g(\mu)_i = (i-1) + 1 + x_i - 1 = x_i + i - 1$$

$$g(\mu)_{i+1} = (i-1) + 1 + 1 + x_{i+1} - 1 = x_{i+1} + i$$

Know: $2x_i - 1 = \mu_i > \mu_{i+1} = 2x_{i+1} - 1$

$$\Rightarrow x_i > x_{i+1}$$

$$x_i + i - 1 > x_{i+1} + i - 1 \Rightarrow \underset{\substack{\| \\ g(\mu)_i}}{x_i + i - 1} \geq \underset{\substack{\| \\ g(\mu)_{i+1}}}{x_{i+1} + i}.$$