falling factorial $(n)_k = n\cdot(n-1)\cdots(n-k+1)$

$$n! = (n)_k (n-k)! \quad \text{if } n \geq k$$

Thm. If $|A| = n$, # words of length k w/ no repeated entries is $(n)_k$.

Pf. If $k > n$, no such words exist, also $(n)_k = 0$

otherwise, we have $n! = (n)_k (n-k)!$

Also, # permutations of n things = # ways to pick first k things · # ways to permute remaining $n-k$ things

$$n! \qquad \frac{n!}{(n-k)!} = (n)_k. \qquad (n-k)!$$

□

Thm. # of subsets of n-element set of size k is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Pf. From before:

$$\{\text{subsets of } [n]\} \longleftrightarrow \left\{\begin{matrix}\text{words of length} \\ n \text{ in } \{0,1\}\end{matrix}\right\}$$

$$S \longrightarrow w_S$$

$|S| =$ # times 1 appears in w_S

subsets of size k = # words w/ k 1's = # ways to arrange k 1's and $n-k$ 0's = $\frac{n!}{k!(n-k)!}$ □

Cor. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

Pf. RHS = # subsets of $[n]$

LHS = $\sum_{k=0}^{n}$ # subsets of size k of $[n]$ = # subsets of $[n]$. □

Pascal's identity: If $k \geq 0$, then:

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

$\binom{n}{-1} := 0$

Pf. RHS: # subsets of size k in $[n+1]$

Break up {subsets of size k in $[n+1]$} into two types:

Type I: S contains $n+1$ ($n+1 \in S$)

Type II: S does not contain $n+1$. ($n+1 \notin S$)

(I): {Subsets of size k of $[n+1]$ containing $n+1$} $\longleftrightarrow$ {subsets of size $k-1$ of $[n]$}

# of type I $= \binom{n}{k-1}$

(II): {subsets of size k of $[n+1]$ not containing $n+1$} $\longleftrightarrow$ {subsets of size k of $[n]$}

# of type II $= \binom{n}{k}$

$$\Rightarrow \binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}. \qquad \square$$

Multisets: Given set S, a multiset from S is like a subset but we allow repetitions.

size is total number of elements (counting repetitions)

Ex Multisets of $[3]$ of size 3:

111 112 113 122 123

133 222 223 233 333

{multisets of $[3]$ of size 3}

↓

Add 1 to second element

Add 2 to third element:

123 124 125 134 135

145 234 235 245 345

{subsets of $[5]$ size 3}

Thm. # of k-element multisets of n-element set is

$$\binom{n+k-1}{k}.$$

Pf.

$$\left\{\begin{matrix}\text{multisets of } [n] \\ \text{of size } k\end{matrix}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{matrix}\text{subsets of } [n+k-1] \\ \text{of size } k\end{matrix}\right\}.$$

ψ S

f: Write elements of S in increasing order:

$$s_1 \leq s_2 \leq \cdots \leq s_k$$

$$f(S) = \{ s_1, s_2+1, s_3+2, \ldots, s_k+(k-1) \}.$$

Note: $s_1 < s_2+1 < s_3+2 < s_4+3 < \cdots < s_k + (k-1) \leq n+k-1$

Hence $f(S)$ in fact has k-elements.

Given T subset of size k in $[n+k-1]$

list elements $t_1 < t_2 < \cdots < t_k$ of T

$$g(T) = \{ t_1, t_2-1, t_3-2, \ldots, t_k-(k-1) \}.$$

Note: $1 \leq t_1 \leq t_2-1 \leq t_3-2 \leq \cdots \leq t_k-(k-1) \leq n$

g, f are inverse to each other, hence

$$\#\begin{matrix}\text{multisets of } [n] \\ \text{of size } k\end{matrix} = \#\begin{matrix}\text{subsets of } [n+k-1] \\ \text{of size } k\end{matrix} = \binom{n+k-1}{k}. \quad \square$$

Poker hands   52 cards = (value, suit)

4 suits: ♣, ♡, ♤, ♢

13 values: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K

Choose subset of size 5 w/ special properties:

① (4 of a kind) 4 of cards have same value
5th one: no condition.

data: value of these 4 cards ← 13
+ extra card. ← 48

answer: $13 \cdot 48$

② (Full house): 3 cards same value
other 2 cards also same value

choices for values: $(13)_2 = 13 \cdot 12$
choices for suits for triple: $\binom{4}{3} = 4$
choices for suits for pair: $\binom{4}{2} = 6$

answer: $13 \cdot 12 \cdot \binom{4}{3}\binom{4}{2}$

③ (Two pairs) 2 cards of same value
2 other cards of same value
1 more card

} 3 different values

choices for values of pairs is $\binom{13}{2}$
choices for value of extra card is 11

$= \binom{13}{2} 11 = \frac{13 \cdot 12 \cdot 11}{2}$

---

alternative [
Choices for value of extra card is 13
Choices for values of pairs is $\binom{12}{2}$
]

$13 \cdot \binom{12}{2} = \frac{13 \cdot 12 \cdot 11}{2}$

---

choices for suits for pairs: $\binom{4}{2}^2$
choices for suit for extra: 4

$\binom{4}{2}^2 4$

answer: $\binom{13}{2} \cdot 11 \cdot \binom{4}{2}^2 \cdot 4$

(4) (straight). values can be put in consecutive order (no condition on suits)

(funny rule: A is before 2 or after K.)

choices for values: choose smallest value : 10

choices for suits: 4^5

answer: $10 \cdot 4^5$