Strong induction

Strategy: ① Prove $P(0)$

② Use $P(0), P(1), \dots, P(n)$ to prove $P(n+1)$

Ex. Want to prove every polynomial in X is a linear combination of powers of $x-1$:

$1, (x-1), (x-1)^2, (x-1)^3, \dots$

let $P(n)$ be "every polynomial of degree n is linear combination of powers of $(x-1)$"

① $P(0)$: every degree 0 polynomial is a constant c,

$c = c \cdot 1$ ✓

② Let $f(x)$ be polynomial of degree $n+1$.

Let α be leading coefficient of $f(x)$.

$f(x) - \alpha(x-1)^{n+1}$ has degree $\le n$

by induction $\Rightarrow$ $f(x) - \alpha(x-1)^{n+1}$ is linear combination of powers of $(x-1)$.

$\square$

$$f(x) = \alpha x^{n+1} + \boxed{\text{smaller than degree } n}$$

$$-\alpha(x-1)^{n+1} = -(\alpha x^{n+1} + \boxed{\text{smaller}})$$

$$f(x) - \alpha(x-1)^{n+1} = \text{smaller than deg } n$$

$$= \sum \alpha_i (x-1)^i$$

$$\Rightarrow f(x) = \alpha(x-1)^{n+1} + \sum \alpha_i (x-1)^i$$

Permutations Def. S = set. permutation is an ordering of elements of S.

Ex. $S = \{1,2,3\}$, permutations are: (there are 6)

123   132   213   231   312   321

Def. $0! = 1$. if n positive integers $n! = n \cdot (n-1)!$

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

$0! = 1$, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, $5! = 120$, $6! = 720$

Thm. If $|S| = n$, (assume $n>0$) then there are $n!$ permutations of S.

Pf. Induction on n.

$n=1$: one thing is ordered in unique way. $1! = 1$ ✓

Assume known for n.

Let S be set of size $n+1$. To get a permutation,

Pick something to be first (n+1 choices)

Need to order remaining n elements ($n!$ ways by induction)

$\Rightarrow$ $\underbrace{(n+1) \cdot n!}_{(n+1)!}$ many ways to order everything ✓ □

Ex. ① 2 Red flowers, 1 black flower

$R_1 R_2 B$ $R_1 B R_2$ $B R_1 R_2$

$R_2 R_1 B$ $R_2 B R_1$ $B R_2 R_1$ ← duplicates

answer is: 3 ways to arrange.

$$3 = \frac{3!}{2!}$$

(2) 10 red flowers, 5 black flowers

naively: $15!$ permutations.

redundant info: ordering of red flowers $10!$

ordering of black flowers $5!$

$\Rightarrow \dfrac{15!}{10!\,5!}$ many ways if flowers of same color are treated same.

(3) r red flowers, b black flowers:

$$\frac{(r+b)!}{r!\,b!}$$

(4) also: g green flowers: $\dfrac{(r+b+g)!}{r!\,b!\,g!}$

Total # of permutations $=$ $\boxed{(\text{\# arrangements})}$ $\cdot$ (# ways to order flowers of same color)

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n objects, each one has a "type" (color)

k possible types, a_i objects of type i.

Thm. # ways to arrange objects (treating same type as identical) is

$$\frac{n!}{a_1!\,a_2!\,\cdots\,a_k!}$$

Def. Multinomial coefficient:

if $a_1+\cdots+a_k=n$, then $\dbinom{n}{a_1, a_2, \ldots, a_k} = \dfrac{n!}{a_1!\,a_2!\cdots a_k!}$

Special case ($k=2$): $\binom{n}{a_1}$ means $\binom{n}{a_1, a_2}$

binomial coefficient

WORDS. Def. Let A be a set (alphabet)

A word is a finite sequence of elements from A.

length is number of entries (could be 0)

Ex. $A = \{a, b\}$ words of length ≤ 2:

$\emptyset$, a, b,

aa, ab, ba, bb

Thm. If $|A| = n$, then there are n^k words of length k.

Pf. Words of length k are k-tuples of elements of A, i.e., elements of $A^k = \underbrace{A \times A \times \cdots \times A}_{k}$ $\square$

$\boxed{[n] := \{1, \ldots, n\} \text{ for integer } n \geq 0.}$

Ex. We saw that #subsets of $[n] = 2^n$

Let $A = \{0, 1\}$

Given a subset $S \subseteq [n]$, define w_S as follows:

if $i \in S$, then ith entry of w_S is 1

$i \notin S$, then ith entry of w_S is 0

This gives $f: \left\{ \begin{array}{c} \text{subsets of} \\ [n] \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{words in } \{0,1\} \\ \text{of length } n \end{array} \right\}$

$$f(S) = w_S$$

Define inverse g as follows: given word w, define $g(w) \subseteq [n]$ by $g(w) = \{\text{indices } i \text{ s.t. } w_i = 1\}$

f, g inverses of each other: so bijections

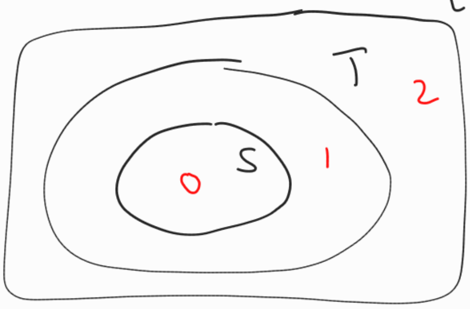
$\Rightarrow$ # subsets of $[n]$ = # words in $\{0,1\}$ of length n = 2^n

EX. How many choices of $S, T \subseteq [n]$ s.t. $S \subseteq T$?

Let $A = \{$ "in S and T", "in T but not S", "not in T or S" $\}$

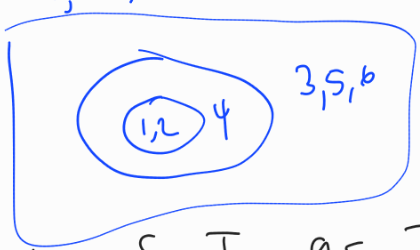
0 1 2

$[n]$



$n = 6$

$S = \{1,2\}$, $T = \{1,2,4\}$



Given word in $\{0,1,2\}$, get S, T as follows:

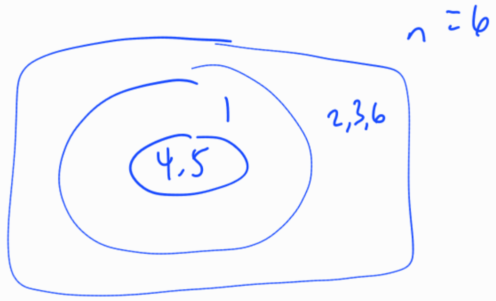
$S = \{\text{indices where } 0 \text{ appears}\}$

$T = \{\text{indices where } 0 \text{ or } 1 \text{ appears}\}$

$$\Rightarrow f: \left\{ \begin{array}{c} \text{words in } \{0,1,2\} \\ \text{of length } n \end{array} \right\} \underset{g}{\overset{}{\rightleftarrows}} \{ S \subseteq T \subseteq [n] \}$$

$g(S \subseteq T)$ is word whose ith entry is $\begin{cases} 0 & \text{if } i \in S \\ 1 & \text{if } i \in T \setminus S \\ 2 & \text{if } i \notin T \end{cases}$

$\Rightarrow$ bijection $\Rightarrow$ answer is 3^n



$S = \{4,5\}$

$T = \{1,4,5\}$