$A$ = alphabet of size $k$.

Count words of length $n$ up to rotation (cyclic symmetry)

following are same:

**Example:**

$a_1 a_2 a_3 a_4$ $\quad$ $a_3 a_4 a_1 a_2$

$a_2 a_3 a_4 a_1$ $\quad$ $a_4 a_1 a_2 a_3$

**Naive guess:** $\dfrac{\#\text{words of length } n}{n} = \dfrac{k^n}{n}$ (wrong)

**Refinement:** Note that $0101$ has only 2 different rotations:

$1010$

Given word $w$, its period is least number of rotations of $w$ needed to get $w$ back.

**Ex.** period of $0101$ is 2.

Let $\omega(d)$ = # of words of period $d$

**Observations:** • period has to divide length of word.

• $\omega(d)$ does not depend on $n$

**Def.** Equivalence class of word of length $n$ upto cyclic symmetry is necklace of length $n$.

$$\boxed{\#\text{necklaces of length } n = \sum_{d \mid n} \frac{\omega(d)}{d}}$$

**Ex.** # necklaces of length 4

$= \omega(1) + \dfrac{\omega(2)}{2} + \dfrac{\omega(4)}{4}$

**Second identity:** $\underset{\substack{\| \\ k^n}}{\#\text{words of length } n} = \sum_{d \mid n} \omega(d)$

By using this identity, can try to solve for $\omega(d)$:

Ex. Solve for $\omega(4)$:

Start w/
$$k^4 = \omega(1) + \omega(2) + \omega(4)$$
$$k^2 = \omega(1) + \omega(2)$$

---

$$k^4 - k^2 = \omega(4)$$

Solve for $\omega(6)$:

$$k^6 = \omega(1) + \omega(2) + \omega(3) + \omega(6)$$
$$k^3 = \omega(1) + \omega(3)$$
$$k^2 = \omega(1) + \omega(2)$$
$$k = \omega(1)$$

---

$$\omega(6) = k^6 - k^3 - k^2 + k$$

How to get general equation for $\omega(n)$?

Möbius function

Def. Define $\mu(1) = 1$. For $n > 1$,

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by } p^2 \text{ for any prime } p \\ (-1)^r & \text{if } n \text{ is a product of } r \text{ different primes.} \end{cases}$$

Ex.

$\mu(1) = 1$

$\mu(2) = -1$

$\mu(3) = -1$

$\mu(4) = 0$ (since $2^2$ divides 4)

$\mu(5) = -1$

$\mu(6) = (-1)^2 = 1$

$\mu(7) = -1$

$\mu(8) = 0$ (since $2^2$ divides 8)

---

$\mu(9) = 0$

$\mu(10) = (-1)^2 = 1$

$\mu(11) = -1$

$\mu(12) = 0$

$12 = 2^2 \cdot 3$

Lemma. Let $n > 1$. Then $\sum_{d \mid n} \mu(d) = 0$

Pf. Write prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$ ($a_i \geq 1$, $p_i$'s different primes)

$$\sum_{d \mid n} \mu(d) = \sum_{\substack{0 \le e_1 \le a_1 \\ 0 \le e_2 \le a_2 \\ \vdots \\ 0 \le e_r \le a_r}} \mu(p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}) = \sum_{\substack{0 \le e_1 \le 1 \\ 0 \le e_2 \le 1 \\ \vdots \\ 0 \le e_r \le 1}} \mu(\underbrace{p_1^{e_1} \cdots p_r^{e_r}}_{(-1)^{\#(e_i \text{ equal to } 1)}})$$

$$= \sum_{S \subseteq \{p_1, \ldots, p_r\}} \mu(\underbrace{\prod_{i \in S} i}_{(-1)^{|S|}}) = \sum_{S \subseteq \{p_1, \ldots, p_r\}} (-1)^{|S|} = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \underset{r \geq 1.}{=} 0 \qquad \square$$

---

Thm (Möbius inversion). Let $\alpha, \beta$ be complex-valued functions defined on positive integers.

① If $\alpha(d) = \sum_{e \mid d} \beta(e)$ for all $d$,

Then $\beta(d) = \sum_{e \mid d} \mu\left(\frac{d}{e}\right) \alpha(e)$ for all $d$.

② If $\alpha(d) = \prod_{e \mid d} \beta(e)$ for all $d$

and $\beta(e) \neq 0$ for all $e$, Then

$\beta(d) = \prod_{e \mid d} \alpha(e)^{\mu(d/e)}$ for all $d$.

Pf of ①. $\displaystyle\sum_{e|d} \mu\left(\frac{d}{e}\right) \alpha(e) = \sum_{e|d} \mu\left(\frac{d}{e}\right) \sum_{f|e} \beta(f)$

$$= \sum_{\substack{(d,e,f) \\ f|e|d}} \mu\left(\frac{d}{e}\right) \beta(f) = \sum_{f|d} \beta(f) \sum_{\substack{e \text{ s.t.} \\ f|e \,\&\, e|d}} \mu\left(\frac{d}{e}\right)$$

Have bijection:

$$\boxed{\begin{array}{ccc} \{e \mid f|e \,\&\, e|d\} & \longleftrightarrow & \{r \mid r \text{ divides } \frac{d}{f}\} \\ e & \longrightarrow & \frac{d}{e} \end{array}}$$

Write prime factorization $d = p_1^{a_1} \cdots p_r^{a_r}$

$f = p_1^{b_1} \cdots p_r^{b_r}$

Since $f|d$, $b_i \le a_i$ for all $i$.

If $f|e$ & $e|d$, then $e = p_1^{c_1} \cdots p_r^{c_r}$

where $b_i \le c_i \le a_i$ for all $i$.

---

Then $\frac{d}{e} = p_1^{a_1-c_1} p_2^{a_2-c_2} \cdots p_r^{a_r-c_r}$.

$\frac{d}{f} = p_1^{a_1-b_1} p_2^{a_2-b_2} \cdots p_r^{a_r-b_r}$.

$$\begin{array}{ccc} \{e \mid f|e \,\&\, e|d\} & & \{x \mid x \text{ divides } d/f\} \\ \Updownarrow & & \Updownarrow \\ \left\{(c_1,\dots,c_r) \;\middle|\; \begin{array}{l} b_1 \le c_1 \le a_1 \\ b_2 \le c_2 \le a_2 \\ \vdots \\ b_r \le c_r \le a_r \end{array}\right\} & \longrightarrow & \left\{(e_1,\dots,e_r) \;\middle|\; \begin{array}{l} 0 \le e_1 \le a_1-b_1 \\ 0 \le e_2 \le a_2-b_2 \\ \vdots \\ 0 \le e_r \le a_r-b_r \end{array}\right\} \\ (c_1,\dots,c_r) & \longrightarrow & (c_1-b_1,\dots,c_r-b_r) \end{array}$$

$$\sum_{f|d} \beta(f) \sum_{\substack{e \text{ s.t.} \\ f|e \,\&\, e|d}} \mu\left(\frac{d}{e}\right) = \sum_{f|d} \beta(f) \underbrace{\sum_{x | \frac{d}{f}} \mu(x)}_{\text{if } \frac{d}{f} > 1, \text{ sum is } 0.}$$

$$= \beta(d) \sum_{x|1} \mu(1) = \beta(d) \qquad \square$$

---

For our problem: let $\beta = \omega$.

$$\alpha(d) = k^d$$

Then $\alpha(d) = \sum_{e|d} \beta(e)$

$\underline{\text{Möbius}} \Longrightarrow$ $\beta(d) = \sum_{e|d} \mu\left(\frac{d}{e}\right) \alpha(e)$

<u>Cor.</u> For any positive integer $d$,

$$\omega(d) = \sum_{e|d} \mu\left(\frac{d}{e}\right) k^e$$

---

<u>EX.</u> # necklaces of length 4:

$\omega(1) = \sum_{e|1} \mu\left(\frac{1}{e}\right) k^e = \mu(1) k = k$

$\omega(2) = \mu\left(\frac{2}{1}\right) k^1 + \mu\left(\frac{2}{2}\right) k^2 = \mu(2) k + \mu(1) k^2 = -k + k^2$

$\omega(4) = \mu\left(\frac{4}{1}\right) k^1 + \mu\left(\frac{4}{2}\right) k^2 + \mu\left(\frac{4}{4}\right) k^4$

$= \mu(4) k + \mu(2) k^2 + \mu(1) k^4 = k^4 - k^2$

#necklaces of length $4 = \omega(1) + \frac{\omega(2)}{2} + \frac{\omega(4)}{4} = k + \frac{k^2 - k}{2} + \frac{k^4 - k^2}{4}$

$$\omega(6) = \mu\left(\tfrac{6}{1}\right)k^1 + \mu\left(\tfrac{6}{2}\right)k^2 + \mu\left(\tfrac{6}{3}\right)k^3 + \mu\left(\tfrac{6}{6}\right)k^6$$
$$= k - k^2 - k^3 + k^6$$

---

<u>Notation</u>: $i = \sqrt{-1}$

Euler's identity: $e^{2\pi i} = 1$.

$\Longrightarrow$ Solutions to $x^n - 1 = 0$ are $e^{2\pi i/n}, e^{2\pi i \cdot 2/n}, \ldots, e^{2\pi i(n-1)/n}$

nth roots of unity.

If $k, n$ have common factor $r$, then $e^{2\pi i k/n}$ is also $\left(\frac{n}{r}\right)^{th}$ root of unity.

If $k, n$ are relatively prime, then $e^{2\pi i k/n}$ is called primitive nth root of unity.

---

<u>Def</u>. nth cyclotomic polynomial is

$$\Phi_n(x) = \prod_k \left(x - e^{2\pi i k/n}\right)$$

$\leftarrow$ $0 \le k \le n-1$ & $k, n$ relatively prime.

By definition, $x^n - 1 = \prod_{d|n} \Phi_d(x)$

<u>Möbius inversion</u>: $\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}$

EX. $n=6$: $\Phi_6(x) = \prod_{d|6} (x^d - 1)^{\mu(6/d)} = \dfrac{(x^6-1)(x-1)}{(x^2-1)(x^3-1)}$

$$= x^2 - x + 1.$$

$n=8$: $\Phi_8(x) = \prod_{d|8} (x^d-1)^{\mu(8/d)} = \dfrac{x^8-1}{x^4-1} = x^4+1$

---

Crazy dice:

Roll 2 6-sided dice, distribution for sum:

| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 5 | 4 | 3 | 2 | 1 |

Question: Can we relabel sides of dice so that distribution of sum is as above?

Constraint: must use positive integers.

One other solution:

| | 1 | 3 | 4 | 5 | 6 | 8 |
|---|---|---|---|---|---|---|
| 1 | 2 | 4 | 5 | 6 | 7 | 9 |
| 2 | 3 | 5 | 6 | 7 | 8 | 10 |
| 2 | 3 | 5 | 6 | 7 | 8 | 10 |
| 3 | 4 | 6 | 7 | 8 | 9 | 11 |
| 3 | 4 | 6 | 7 | 8 | 9 | 11 |
| 4 | 5 | 7 | 8 | 9 | 10 | 12 |