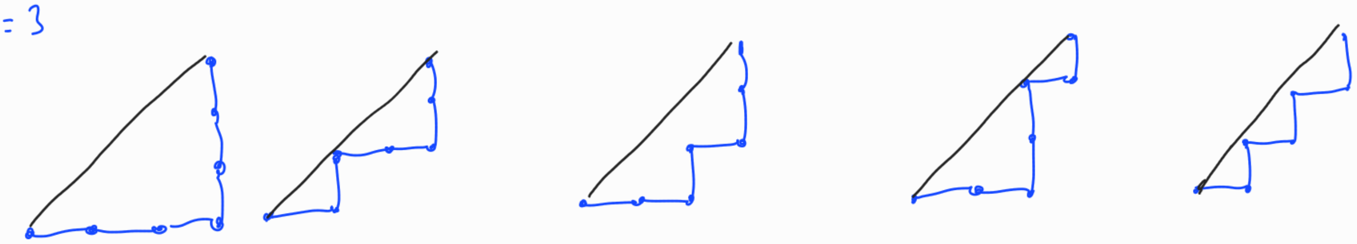
Catalan numbers

C_n = # "forward paths" from $(0,0)$ to (n,n) which never go above line $x=y$.

$n=3$



bijection w/ parentheses: right step $\leftrightarrow$ (
up step $\leftrightarrow$)

Composition of OGF.

a_n = # structures (call it α) on a set $\{1,\dots,n\}$.

Assume $a_0 = 0$.

Define h_n = # ways to break $[n]$ into disjoint, nonempty, consecutive pieces and place structure α on each piece.

Define $A(x) = \sum_{n\geq 0} a_n x^n$, $H(x) = \sum_{n\geq 0} h_n x^n$

$h_0 = 1$ by convention

Thm. $$H(x) = \frac{1}{1-A(x)} = \sum_{k\geq 0} A(x)^k$$

Pf. What is $[x^n]A(x)^k$?

This is # ways to break $[n]$ into k disjoint nonempty consecutive pieces and put structure α on each.

$$\Rightarrow h_n = \sum_{k\geq 0}[x^n]A(x)^k \Rightarrow H(x) = \sum_{k\geq 0} A(x)^k. \qquad \square$$

Ex. n soldiers lined up.

want to assign them to teams (lazy way by picking consecutive intervals in the line) and picking leader for each team.

h_n = # ways to make this choice.

Structure α = way to pick leader
$a_n = n$ (note $a_0 = 0$, so previous thm applies)

$A(x) = \sum_{n \geq 0} a_n x^n$, then $H(x) = \sum_{n \geq 0} h_n x^n = \frac{1}{1 - A(x)}$.

$$A(x) = \sum_{n \geq 0} n x^n = \frac{x}{(1-x)^2}$$

$$\Rightarrow H(x) = \frac{1}{1 - \frac{x}{(1-x)^2}} = \frac{(1-x)^2}{(1-x)^2 - x} = \frac{1 - 2x + x^2}{1 - 3x + x^2}.$$

could finish by partial fractions

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general composition $B(A(x)) = \sum_{k \geq 0} b_k A(x)^k$

b_k = # ways to put structure β on set of k things

h_n = # ways to break $[n]$ into consecutive nonempty disjoint intervals, put structure α on each interval, put structure β on set of intervals.

$H(x) = \sum_{n \geq 0} h_n x^n$

Thm. $H(x) = B(A(x))$.

Ex. n soldiers as before, pick teams & leaders.

+ some subset of teams selected to do night surveillance.

$a_n = n$ as before

$b_k =$ # ways to choose subset $= 2^k$

$h_n =$ # ways to do this

$$A(x) = \sum_{n\geq 0} a_n x^n, \quad B(x) = \sum_{k\geq 0} b_k x^k, \quad H(x) = \sum_{n\geq 0} h_n x^n$$

then $H(x) = B(A(x))$.

$$A(x) = \frac{x}{(1-x)^2}, \quad B(x) = \sum_{k\geq 0} 2^k x^k = \frac{1}{1-2x}$$

$$H(x) = \frac{1}{1 - \frac{2x}{(1-x)^2}} = \frac{1-2x+x^2}{1-4x+x^2}$$

$\square$

Exponential generating functions (EGF)

(a_n) sequence. Its EGF is FPS

$$A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n.$$

(Rmk). If $a_n = 1$ for all n, then

$$A(x) = \sum_{n\geq 0} \frac{x^n}{n!} = e^x = \exp(x)$$

Ex. $a_0 = 1$, $a_n = n(a_{n-1} - n + 2)$ for $n \geq 1$.

Set $B(x) = \sum_{n\geq 0} a_n x^n$.

$$B(x) = 1 + \sum_{n\geq 1} a_n x^n = 1 + \sum_{n\geq 1} n(a_{n-1} - n + 2)\, x^n$$

$$= 1 + \sum_{n\geq 1} n a_n x^n - \underline{\sum_{n\geq 1} n(n-2)\, x^n}$$

?

$$x B'(x) = x \sum_{n\geq 0} n a_n x^{n-1} = \sum_{n\geq 0} n a_n x^n$$

Try EGF instead:

$$\boxed{A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n.}$$

$$A(x) = 1 + \sum_{n\geq 1} \frac{a_n}{n!} x^n = 1 + \sum_{n\geq 1} \frac{n(a_{n-1} - n + 2)}{n!} x^n$$

$$= 1 + x \sum_{n\geq 1} \frac{a_{n-1}}{(n-1)!} x^{n-1} - \sum_{n\geq 1} \frac{(n-2)}{(n-1)!} x^n$$

$$= 1 + x A(x) - \sum_{n\geq 1} \frac{n-2}{(n-1)!} x^n$$

$$= 1 + x A(x) - \sum_{n\geq 2} \frac{n-1}{(n-1)!} x^n + \sum_{n\geq 1} \frac{1}{(n-1)!} x^n$$

$$= 1 + x A(x) - x^2 \sum_{n\geq 2} \frac{x^{n-2}}{(n-2)!} + x \sum_{n\geq 1} \frac{x^{n-1}}{(n-1)!}$$

$$= 1 + x A(x) - x^2 e^x + x e^x$$

$$A(x) - x A(x) = 1 - x^2 e^x + x e^x$$

$$A(x) = \frac{1 - x^2 e^x + x e^x}{1 - x} = \frac{1 + x(1-x) e^x}{1 - x}$$

$$= \frac{1}{1-x} + xe^{x}$$

$$= \boxed{\sum_{n\geq 0} x^n + \sum_{n\geq 0} \frac{x^{n+1}}{n!}}$$

coeff of x^n is $\boxed{1 + \frac{1}{(n-1)!}}$ $n \geq 1$

Coeff of x^n in $A(x) = \boxed{\frac{a_n}{n!}}$

$$\Rightarrow \boxed{a_n = n! + n}$$

Lemma. $A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n$, $B(x) = \sum_{n\geq 0} \frac{b_n}{n!} x^n$

Set $C(x) = A(x)B(x) = \sum_{n\geq 0} \frac{c_n}{n!} x^n$. Then

$$\boxed{c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}.}$$

Pf. $[x^n]\, C(x) = \frac{c_n}{n!}$

$$[x^n]\, C(x) = [x^n]\, A(x)B(x) = \sum_{i=0}^{n} \frac{a_i}{i!} \frac{b_{n-i}}{(n-i)!}$$

$$\Rightarrow c_n = n! \sum_{i=0}^{n} \frac{a_i b_{n-i}}{i!(n-i)!} = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}. \qquad \square$$

Suppose a_n = # structures of type α on set of size n

b_n = # structures of type β on set of size n.

Define c_n = # ways to pick subset $S \subseteq [n]$ and put structure α on S, structure β on $[n] \setminus S$.

Define $A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n$, $B(x) = \sum_{n\geq 0} \frac{b_n}{n!} x^n$

$$C(x) = \sum_{n\geq 0} \frac{c_n}{n!} x^n.$$

<u>Thm</u>. $C(x) = A(x)\, B(x)$.

<u>EX</u>. n football players.

Split into 2 groups,

order first group,

order second group + choose one of 3 colors for each person's uniform.

How many ways to do this? let c_n be answer.

a_n = #ways to order n things $= n!$

b_n = #ways to order n things and choose color for each thing $= 3^n n!$

Define $A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n = \sum_{n\geq 0} x^n = \frac{1}{1-x}$.

$B(x) = \sum_{n\geq 0} \frac{b_n}{n!} x^n = \sum_{n\geq 0} 3^n x^n = \frac{1}{1-3x}$

$C(x) = \sum_{n\geq 0} \frac{c_n}{n!} x^n. = \frac{1}{(1-x)(1-3x)} = \boxed{\frac{3/2}{1-3x} - \frac{1/2}{1-x}}$

$= \boxed{\frac{3}{2} \sum_{n\geq 0} 3^n x^n - \frac{1}{2} \sum_{n\geq 0} x^n}$

Thm $\Rightarrow C(x) = A(x)\, B(x)$

$\Rightarrow \frac{c_n}{n!} = \frac{3}{2}(3^n) - \frac{1}{2}$

$\Rightarrow c_n = \frac{n!}{2}(3^{n+1} - 1)$

EX. n (distinguishable) telephone polls.

want to paint each one red or blue.

must have even number of blue.

c_n = # ways to do this.

a_n = # ways to paint n polls red subject to our rules $= 1$

b_n = # ways to paint n polls blue subject to our rules

$$= \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}.$$

If $A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n$, $B(x) = \sum_{n\geq 0} \frac{b_n}{n!} x^n$

and $C(x) = \sum_{n\geq 0} \frac{c_n}{n!} x^n$. Then $C(x) = A(x)B(x)$.

$$A(x) = \sum_{n\geq 0} \frac{x^n}{n!} = e^x.$$

$$B(x) = \sum_{\substack{n\geq 0 \\ n \text{ even}}} \frac{x^n}{n!} = \frac{1}{2}\left(\sum_{n\geq 0} \frac{x^n}{n!} + \sum_{n\geq 0} \frac{(-x)^n}{n!}\right) = \frac{e^x + e^{-x}}{2}$$

$$e^x e^x = \left(\sum_{n\geq 0} \frac{x^n}{n!}\right)^2$$

$\overset{?}{=}$ yes, but need to prove

$$e^{2x} = \sum_{n\geq 0} \frac{(2x)^n}{n!}$$

in general, $e^{A(x)} e^{B(x)} = e^{A(x)+B(x)}$

$$C(x) = e^x\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left(e^{2x} + 1\right)$$

$$= \frac{1}{2} \sum_{n\geq 0} \frac{2^n x^n}{n!} + \frac{1}{2}$$

If $n=0$: $\frac{c_0}{0!} = \frac{1}{2} + \frac{1}{2} = 1 \Rightarrow c_0 = 1$

$n>0$, $\frac{c_n}{n!} = \frac{1}{2}\frac{2^n}{n!} = \frac{2^{n-1}}{n!} \Rightarrow c_n = 2^{n-1}$

$$c_n = \begin{cases} 1 & \text{if } n=0 \\ 2^{n-1} & n>0 \end{cases}$$

generalize: say we have k structures $\alpha_1, \ldots, \alpha_k$

w/ respective EGF $A_1(x), \ldots, A_k(x)$

If $C(x) = A_1(x) \cdots A_k(x)$

then $n! \, [x^n] C(x) = \#$ ways to pick k subsets $S_1, \ldots, S_k \subseteq [n]$ (possibly empty) s.t. disjoint and $S_1 \cup \cdots \cup S_k = [n]$ and putting structure α_i on S_i