Math 184, Winter 2022
Homework 5
Due: Friday, Feb. 25 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences.

You may use, without proving it, that for any formal power series $A(x)$ and $B(x)$ without constant terms, we have

$$e^{A(x)+B(x)} = e^{A(x)}e^{B(x)}.$$

(1) For $n > 0$, let a_n be the number of partitions of n such that every part appears at most twice, and let b_n be the number of partitions of n such that no part is divisible by 3. Set $a_0 = b_0 = 1$. Show that $a_n = b_n$ for all n.

The identity $1 + u + u^2 = \frac{1-u^3}{1-u}$ will be helpful.

(2) Let a_n be the sequence which satisfies the recurrence

$$a_n = a_{n-1} + 3\sum_{i=0}^{n-2} a_i a_{n-2-i} \qquad \text{for } n \geq 2$$

and $a_0 = a_1 = 1$. Find a simple formula for $A(x) = \sum_{n\geq 0} a_n x^n$.

(3) (a) We have n distinguishable telephone poles. We want to paint each one either red, blue, green, or black such that an even number of them is red and an odd number of them is blue. How many ways can this be done?

(b) Continuing with that situation, we add the colors white and yellow, but the total number of poles which are white or yellow must be even. How many ways are there to choose colors?

(4) Show that the following two quantities are counted by the nth Catalan number and list/draw the 5 examples when $n = 3$:

(a) The number of paths from $(0, 0)$ to $(2n, 0)$ using the steps $(1, 1)$ and $(1, -1)$ which never go below the x-axis.

(b) Integer partitions $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ such that $\lambda_i \leq n - i$.

(5) (a) Let k be a positive integer. Let $a(k)_n$ be the number of set partitions of $[n]$ into k blocks such that every block has at least 2 elements. Give a simple expression for the exponential generating function

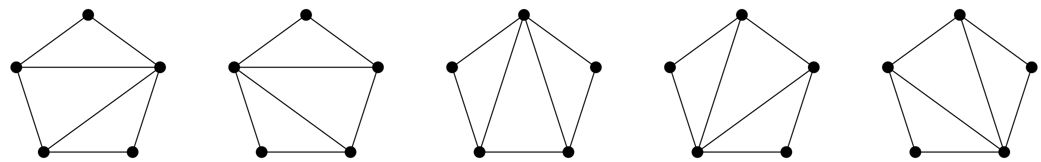
$$A_k(x) = \sum_{n\geq 0} \frac{a(k)_n}{n!} x^n.$$

(b) Let a_n be the number of set partitions of $[n]$ such that every block has at least 2 elements. By convention, $a_0 = 1$. Give a simple expression for the exponential generating function

$$A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n.$$

1. Extra problems (don't turn in)

(6) Let n be a positive integer. Show that the number of ways of triangulating (i.e., drawing diagonals between vertices that do not intersect except at vertices so that the regions are all triangles) a convex polygon with $(n+2)$ vertices is the Catalan number C_n. By convention, the 2-gon has exactly one triangulation and here are the 5 triangulations of a pentagon:



Hint: Fix a vertex v and consider the first vertex (going counterclockwise) that shares a diagonal with it. If none exists, then pick the vertex immediately clockwise of v. This splits the polygon into a left part (where v is no longer on a diagonal) and a right part (which is empty in the exceptional case).

(7) A forward path from $(0,0)$ to (n,n) is **good** if it never goes strictly above the diagonal line $x=y$. Any other forward path is **bad**. From class, the number of good forward paths is the nth Catalan number. In this problem, you will get a new derivation for the formula for Catalan numbers without using generating functions. We denote paths as sequences $(v_1,\dots,v_{2n})$ where each v_i is either the vector $(1,0)$ or $(0,1)$.

(a) Given a bad path $(v_1,\dots,v_{2n})$ from $(0,0)$ to (n,n), let r be the smallest index such that $v_1+\cdots+v_r$ is above the line $x=y$, i.e., the second coordinate is strictly bigger than the first coordinate. Create a new path $(w_1,\dots,w_{2n})$ by

$$w_i = \begin{cases} v_i & \text{if } 1\le i\le r \\ (1,1)-v_i & \text{if } r+1\le i\le 2n \end{cases},$$

i.e., w agrees with v for the first r steps, and we swap all of the remaining steps. Show that w is a forward path from $(0,0)$ to $(n-1,n+1)$.

(b) In (a) we defined a function $\{$bad forward paths from $(0,0)$ to $(n,n)\}\to\{$forward paths from $(0,0)$ to $(n-1,n+1)\}$. Show that this function is a bijection.

(c) Use the formula in HW3, #4(a) and the previous part to get a formula for the number of good forward paths from $(0,0)$ to (n,n).

(8) There are n aisles of shelves in a store. We want to separate them into consecutive nonempty groups for different categories of items. In addition, each category will be painted either red, blue, or green, and we will select some nonempty subset of the categories to be featured in the weekly advertisement. Let h_n be the number of ways to do this. Express $H(x)=\sum_{n\ge 0} h_n x^n$ as a rational function.