Math 184, Winter 2022
Homework 3
Due: Friday, Feb. 4 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences. Some hints on next page.

(1) Fix non-negative integers k, m, n. Consider the identity

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}.$$

  (a) Give an algebraic proof of this identity using that $(x+y)^m(x+y)^n = (x+y)^{m+n}$.
  (b) Give a combinatorial proof of this identity by finding a set whose size can be interpreted as either side of the equation.

(2) Let n be a positive integer. Evaluate the sum

$$\sum_{\substack{0\le i\le n \\ i \text{ even}}} i\binom{n}{i}2^{n-i}.$$

(3) Prove the following identities about the number of integer partitions:
  (a) For $n \ge k$, $p_k(n) = p_{\le k}(n-k)$.
  (b) For $n > 0$, the number of partitions of n not using 1 as a part is $p(n) - p(n-1)$.

(4) A "forward path" in the plane is a sequence of steps of the form $(1,0)$ and $(0,1)$.
  (a) Let a, b be non-negative integers. How many forward paths are there from $(0,0)$ to (a,b)?
  (b) Let $S_{a,b}$ be the set of integer partitions λ such that $\ell(\lambda) \le b$ and $\lambda_1 \le a$. Find a bijection between $S_{a,b}$ and the set of forward paths from $(0,0)$ to (a,b).

(5) Let d be a positive integer. Prove the following identity of formal power series

$$\left(\sum_{a\ge 0} x^a\right)^d = \sum_{n\ge 0}\binom{d+n-1}{n}x^n.$$

Note: this follows from the general binomial theorem, which is not proven in this class; don't use it here. Just use the definition of multiplication of formal power series.

1. Extra practice problems (don't turn in)

(6) Evaluate the following sums:

(a) $\displaystyle\sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i}$

(b) $\displaystyle\sum_{i=0}^{n} i^2 \binom{n}{i} 3^i$

(c) $\displaystyle\sum_{\substack{0 \le i \le n \\ i \text{ odd}}} i \binom{n}{i} = \sum_{\substack{0 \le i \le n \\ i \text{ even}}} i \binom{n}{i}$

(7) Let $a_1, \dots, a_d$ be non-negative integers. Generalize the definition of forward path to d dimensions by using steps which increase one of the coordinates by 1, i.e., using the steps

$$(1, 0, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, 0, 0, \dots, 1).$$

How many forward paths are there from $(0, 0, \dots, 0)$ to $(a_1, a_2, \dots, a_d)$?

(8) Let n and k be positive integers. Evaluate the sum

$$\sum_{(a_1,\dots,a_k)} a_1 \cdots a_k \binom{n}{a_1, \dots, a_k}$$

where the sum is over all compositions of n into k parts.

2. Hints

2: First evaluate the sums

$$\sum_{i=0}^{n} i \binom{n}{i} (-1)^i 2^{n-i}, \qquad \sum_{i=0}^{n} i \binom{n}{i} 2^{n-i}.$$

4b: Think of a forward path as splitting the $b \times a$ rectangle into two pieces.

5: Interpret all the ways of multiplying terms in $(\sum_{a \ge 0} x^a)^d$ as weak compositions.