Artinian modules

A = ring, M = A-module

Def. M is artinian if it satisfies descending chain condition (DCC): any chain $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ stabilizes: $\exists n$ s.t. $M_n = M_{n+1} = M_{n+2} = \cdots$

Equivalently, M is artinian if every nonempty set of submodules of M has a minimal element.

A is an artinian ring if artinian as A-module, i.e., ideals satisfy DCC.

Prop. Given short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, M_2 is artinian $\iff$ M_1 & M_3 are artinian.

Cor. Finite direct sums of artinian are artinian. If A is artinian ring, every f.g. A-module is artinian.

Def. • An A-module M is simple if every submodule is either $\{0\}$ or M.

• If $M \neq 0$, M has a composition series if $\exists$ chain of submodules $M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$ s.t.

M_i / M_{i+1} is simple for all i.

The length of this chain is n. Define $\ell(M)$ to be minimal length of any composition series of M (if it exists). Define $\ell(M) = \infty$ if no composition series exists.

Define $\ell(0) = -1$. M is finite length module if $\ell(M) < \infty$.

Ex. If A is a field, then A artinian.

A vector space V is simple $\iff$ $\dim V \leq 1$

V has a composition series $\iff$ $\dim V < \infty$, $\ell(V) = \dim V$.

Prop. $M \neq 0$, $\ell(M) < \infty$.

① If $N \subsetneq M$, then $\ell(N) < \ell(M)$ & $\ell(M/N) < \ell(M)$.

② Every strict chain of submodules of M has length $\leq \ell(M)$.
Every composition series of M has length exactly $\ell(M)$

③ Every strict chain of submodules of M can be completed to a composition series by inserting more submodules.
In particular, every strict chain of length $\ell(M)$ is a composition series.

Pf. Choose min. length composition series $0 = M_n \subsetneq M_{n-1} \subsetneq \cdots \subsetneq M_0 = M$
set $N_i = N \cap M_i$. We have injective maps $N_i/N_{i+1} \to M_i/M_{i+1}$

Since M_i/M_{i+1} is simple, image is either 0 or all of M_i/M_{i+1}.

If 0: $N_i = N_{i+1}$
Else, N_i/N_{i+1} nonzero and simple

So deleting N_{i+1} whenever $N_i = N_{i+1}$, we obtain composition series of N from $0 = N_n \subseteq N_{n-1} \subseteq \cdots \subseteq N_0 = N$

So $\ell(N) \leq \ell(M)$.
Note: if equality, by reverse induction we conclude that $N_i = M_i$ for all i.
$\Rightarrow N = M$.

Showing $\ell(M/N) < \ell(M)$ is similar.

② let $0 = M_k' \subsetneq \cdots \subsetneq M_0' = M$ be any strict chain.

By ①, $\ell(M) > \ell(M_1') > \ell(M_2') > \cdots > \ell(M_k') = -1$

$\Rightarrow \ell(M) \geq k$.

$\Rightarrow$ Every composition series has length $\leq \ell(M)$

hence length exactly $\ell(M)$ by definition.

③ Given $N \subsetneq N'$, if N'/N is not simple, then $\exists P \subsetneq N'/N$

s.t. $P \neq 0$, let P' = preimage of P under $N' \to N'/N$.

Then have $N \subsetneq P' \subsetneq N'$.

So any non composition series can be made longer to get another strict chain.

This process terminates since length can't exceed $\ell(M)$.

If length of strict chain is $\ell(M)$, can't increase by ②, so it was already a composition series. $\square$

<u>Cor.</u> $\ell(M) < \infty \iff M$ is noetherian <u>and</u> artinian.

<u>Pf.</u> • If $\ell(M) < \infty$, then every strict chain is finite length, so M satisfies DCC & ACC.

• Conversely, suppose M is noeth & artinian (and $M \neq 0$).

Consider set of proper submodules of M (nonempty)

M noeth $\Rightarrow$ this set has maximal element, pick one, call it M_1.

Then M/M_1 is simple by definition. If $M_1 \neq 0$, can repeat to find maximal proper submodule M_2 in M_1

$\Rightarrow$ get chain $M = M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$

Since M is artinian, this process stops after finite number of steps (and last module must be 0)

$\Rightarrow$ M has a composition series. $\square$

Rmk. Noetherian modules are finitely gen. by definition, but artinian modules need not be:

$A = \mathbb{Q}[x] \subset \mathbb{Q}[x, x^{-1}]$ $\leftarrow$ A-module

$M := \mathbb{Q}[x,x^{-1}] / A \cdot x$ (quotient of A-modules)

has basis $\{1, x^{-1}, x^{-2}, \dots\}$ s.t.

$$x \cdot x^{-i} = \begin{cases} x^{-i+1} & \text{if } i \geq 1 \\ 0 & \text{if } i = 0 \end{cases}$$

Then M is artinian, but not finitely generated.

Prop Given short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$,

$$\ell(M_2) = \ell(M_1) + \ell(M_3).$$

pf. First suppose M_1, M_3 both have composition series:

$$0 = N_r \subsetneq N_{r-1} \subsetneq \cdots \subsetneq N_0 = M_1$$

$$0 = P_s \subsetneq P_{s-1} \subsetneq \cdots \subsetneq P_0 = M_3$$

Let P_i' be preimage of P_i under $M_2 \to M_3$.

Then $0 = N_r \subsetneq N_{r-1} \subsetneq \cdots \subsetneq N_0 \subsetneq P_{s-1}' \subsetneq \cdots \subsetneq P_0' = M_2$.

also, $P_i' / P_{i+1}' \cong P_i / P_{i+1}$, so is still simple. comp. series for M_2

$\Rightarrow$ $\ell(M_2) = \ell(M_1) + \ell(M_3)$

If $\ell(M_1) = \infty$ or $\ell(M_3) = \infty$, then we get infinite strict chain for M_2, and hence $\ell(M_2) = \infty$. $\square$