Going up & going down Theorems

Prop. Let $A \subseteq B$ be integral extension of domains. Then

A is a field $\iff$ B is a field.

Pf. Suppose A is a field. Pick $x \in B$ $(x \neq 0)$. $\exists a_1, \dots a_n \in A$ s.t.

$$x^n + a_1 x^{n-1} + \dots + a_n = 0.$$

Pick n minimal so that such an equation exists

$\Rightarrow$ $a_n \neq 0$; If not, then

$$x(x^{n-1} + a_1 x^{n-2} + \dots + a_{n-1}) = 0$$

$$\Rightarrow \quad x^{n-1} + a_1 x^{n-2} + \dots + a_{n-1} = 0$$

$$\Rightarrow -\frac{1}{a_n} x\left(x^{n-1} + a_1 x^{n-2} + \dots + a_{n-1}\right) = 1$$

$$\Rightarrow \frac{1}{x} = -\frac{1}{a_n}\left(x^{n-1} + a_1 x^{n-2} + \dots + a_{n-1}\right) \in B \quad (\text{since } 1/a_n \in A.)$$

$\Rightarrow$ B is a field.

Suppose B is a field. Pick $x \in A$, $x \neq 0$. $1/x \in B$, so is integral over A. $\Rightarrow \exists a_1, \dots a_n \in A$ s.t.

$$\left(\frac{1}{x}\right)^n + a_1\left(\frac{1}{x}\right)^{n-1} + \dots + a_n = 0.$$

Multiply by x^{n-1}:

$$\frac{1}{x} + a_1 + \dots + a_n x^{n-1} = 0$$

$$\frac{1}{x} = -(a_1 + \dots + a_n x^{n-1}) \in A \Rightarrow A \text{ is a field } \square$$

Cor. Let $A \subseteq B$ be integral extension. Let $\mathfrak{q} \subset B$ be prime.

Then $\mathfrak{q}$ is maximal $\iff$ $A \cap \mathfrak{q}$ is maximal in A.

Pf. First, $A/A\cap \mathfrak{q} \subset B/\mathfrak{q}$ is integral.

By previous result, $B/\mathfrak{q}$ is field $\Leftrightarrow$ $A/A\cap\mathfrak{q}$ is a field. $\square$

Thm (Incomparability) let $A \subseteq B$ be integral extension.
let $\mathfrak{p} \subseteq \mathfrak{q}$ be primes in B. If $\mathfrak{p}\cap A = \mathfrak{q} \cap A$, then $\mathfrak{p}=\mathfrak{q}$.

Pf. Let $S = A \setminus (\mathfrak{q}\cap A)$. $\Rightarrow$ $S^{-1}A \subseteq S^{-1}B$ is integral.

Also, $S^{-1}\mathfrak{p} \cap S^{-1}A = S^{-1}\mathfrak{q} \cap S^{-1}A$. These coincide w/ $S^{-1}(\mathfrak{q}\cap A)$
which is maximal in $S^{-1}A$.

$\Rightarrow$ $S^{-1}\mathfrak{p}, S^{-1}\mathfrak{q}$ are maximal ideals of $S^{-1}B$.

But $S^{-1}\mathfrak{p} \subseteq S^{-1}\mathfrak{q}$ $\Rightarrow$ $S^{-1}\mathfrak{p} = S^{-1}\mathfrak{q}$.

By correspondence between primes of $S^{-1}B$ and primes of B not intersecting S, we conclude, $\mathfrak{p}=\mathfrak{q}$. $\square$

Thm (Lying over) let $A\subset B$ be integral extension.
Given prime $\mathfrak{p}\subset A$, $\exists$ prime $\mathfrak{q}\subset B$ s.t. $\mathfrak{q}\cap A = \mathfrak{p}$.

"$\mathfrak{q}$ lies over $\mathfrak{p}$"

I.e., $\boxed{\mathrm{Spec}(B) \to \mathrm{Spec}(A) \text{ is surjective.}}$

Pf. Consider $f: A_{\mathfrak{p}} \to B_{\mathfrak{p}} := B \otimes_A A_{\mathfrak{p}}$, which is integral.

Pick maximal ideal $\mathfrak{m}$ in $B_{\mathfrak{p}}$. Then $f^{-1}(\mathfrak{m})$ is maximal in $A_{\mathfrak{p}}$. $A_{\mathfrak{p}}$ is local $\Rightarrow$ $f^{-1}(\mathfrak{m}) = \mathfrak{p}A_{\mathfrak{p}}$.

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d] & B \arrow[d] & q \\
p & A_p \arrow[r, "f"] & B_p & \supset m \arrow[u] \\
 & pA_p \arrow[ul] & &
\end{tikzcd}$$

Let q = inverse image of m under $B \to B_p$.

Then $q \cap A = p$ $\square$

<u>Thm.</u> (Going up theorem). Let $A \subset B$ be integral extension.

Let $p_1 \subseteq p_2 \subseteq \cdots \subseteq p_n$ be primes in A.

Let $q_1 \subseteq B$ be a prime lying over p_1.

Then $\exists$ $q_1 \subseteq q_2 \subseteq \cdots \subseteq q_n$ s.t. $q_i \cap A = p_i$.

<u>Pf.</u> suffices to consider $n = 2$.

The extension $A/p_1 \subseteq B/q_1$ is integral.

We may as well replace A w/ A/p_1 and assume $p_1 = 0$
B w/ B/q_1 $q_1 = 0$.

Now existence of q_2 follows from lying over thm. $\square$

<u>Lemma</u>. Let $f: A \to B$ ring homomorphism. Let $p \subset A$ prime s.t. $f^{-1}(pB) = p$. Then $\exists$ prime $q \subset B$ s.t. $f^{-1}(q) = p$.

<u>Pf.</u> Let $S = f(A \setminus p)$. Then $f^{-1}(pB) = p \Rightarrow$
$pB \cap S = \emptyset$, so $pS^{-1}B \subset S^{-1}B$ is not unit ideal.

Pick maximal ideal m of $S^{-1}B$ that contains $pS^{-1}B$.

Let q be inverse image of m under $B \to S^{-1}B$.

Then $p \subseteq f^{-1}(q)$. by definition of q.

Suppose $x \notin p$. Then $f(x) \in S$, so $f(x)$ is unit in $S^{-1}B$.
and hence $f(x) \notin q$. $\Rightarrow$ $x \notin f^{-1}(q)$. $\Rightarrow$ $f^{-1}(q) \subseteq p$.

$\Rightarrow$ $f^{-1}(q) = p$. $\square$

Lemma 2. Let A be domain, $K = \mathrm{Frac}(A)$, $\overline{K}$ = alg. closure of K.

① Pick monic polynomials $F \in A[t]$, $G, H \in K[t]$. If $F = GH$, then $G, H \in \tilde{A}[t]$

② Suppose A is normal, pick $x \in \overline{K}$ integral over A.
Then the monic minimal polynomial of x over K has coefficients in A.

Furthermore, if $p \subset A$ is prime and there is some monic polynomial in A w/ non-leading coefficients in p for which $x \in \overline{K}$ is a root, then its monic minimal polynomial over K has non-leading coeff. in p.

Pf. ① $F = GH$. Let $\alpha \in \overline{K}$ be a root of G or H.
$F \in A[t]$ is monic $\Rightarrow$ α is integral over A.
In particular the coeff of G and H are integral over A.
$\Rightarrow$ $G, H \in \tilde{A}[t]$.

② Let F be any integrality equation for x over A.
Let G be monic minimal polynomial of x over K.
By definition, $G \mid F$. ① $\Rightarrow$ $G \in \tilde{A}[t] = A[t]$.

For second part, note $A[t]/pA[t]$ is a domain, and

Let F be a monic polynomial w/ non-leading coeft in p s.t. x is a root. Let G be monic min. poly of x over K.

$\Rightarrow \exists H$ s.t. $F = GH$ & $G, H \in A[t]$.

Reduce mod p: F becomes power of t
$\Rightarrow G, H$ mod p are also powers of t. $\square$

<u>Thm</u> (Going-down thm). Let $A \subseteq B$ be integral extension of domains. Assume A is normal. Let $p_1 \subseteq \cdots \subseteq p_n$ be primes in A. Let q_n be any prime lying over p_n.
<u>Then</u> $\exists\ q_1 \subseteq \cdots \subseteq q_n$ primes in B s.t. $q_i \cap A = p_i$.

<u>Pf</u>. Suffices to handle $n=2$.
Let $S = B \setminus q_2$. Consider ring map $A \to S^{-1}B$.
We're done if we can show that $\boxed{p_1 S^{-1}B \cap A = p_1}$:
by Lemma 1 there would exist prime q' of $S^{-1}B$ whose pullback to A is p_1. Then take q_1 = pullback of q' under $B \to S^{-1}B$.

Pick $x \in p_1 S^{-1}B \cap A$. Write $x = \frac{y}{s}$, $y \in p_1 B$, $s \in B \setminus q_2$.
Need to show $x \in p_1$. Consider A-linear map $\varphi: A[y] \to A[y]$
$\varphi(z) = yz$.

φ takes values in $p_1 A[y]$.

Cayley-Hamilton: φ satisfies monic polynomial equation whose non-leading coeff belong to p_1. $\Rightarrow$ same for y.

Let $K = \operatorname{Frac} A$, $\overline{K}$ = alg. closure of K.

Then $y \in \overline{K}$, by Lemma 2, the monic minimal poly of y over K has non-leading coeff belonging to p_1. Call it F.

Write $F(t) = t^n + a_1 t^{n-1} + \cdots + a_n$, $a_i \in p_1$

Then $$y^n + a_1 y^{n-1} + \cdots + a_n = 0$$

Divide by x^n:

$$\left(\frac{y}{x}\right)^n + \frac{a_1}{x}\left(\frac{y}{x}\right)^{n-1} + \cdots + \frac{a_n}{x^n} = 0$$

$$\Rightarrow \quad s^n + \frac{a_1}{x} s^{n-1} + \cdots + \frac{a_n}{x^n} = 0 . \quad (*)$$

Since $x \in A$, $\frac{a_i}{x^i} \in K$ for all i.

Claim: $(*)$ is minimal polynomial for s over K.

If not, there's a smaller degree one. Multiply by x^n to get smaller degree polynomial for y over K. $\Rightarrow\!\Leftarrow$

Lemma 2 $\Rightarrow$ coeff of $(*)$ belong to A.

so $\frac{a_i}{x^i} \in A$ for all i.

$$\Rightarrow \left(\frac{a_i}{x^i}\right) x^i = a_i \in p_1$$

$$\Rightarrow \frac{a_i}{x^i} \in p_1 \text{ or } x \in p_1$$ we're done

Otherwise $\frac{a_i}{x^i} \in p_1$ for all $i = 1, \dots n$.

$$(*) \Rightarrow s^n = -\left(\frac{a_1}{x} s^{n-1} + \dots + \frac{a_n}{x^n} \right) \in p_1 B \subseteq q_2 .$$

$$\Rightarrow s \in q_2 . \longrightarrow\!\!\!\!\!\!\times\!\!\longleftarrow$$

$\square$

Conclude: $x \in p_1$