Local properties

Prop. Let $M = A$-module. TFAE:

① $M = 0$

② $M_p = 0$ for all primes p.

③ $M_m = 0$ for all maximal ideals m.

Pf. ① $\Rightarrow$ ② $\Rightarrow$ ③

To show ③ $\Rightarrow$ ①, we show: if $M \neq 0$, $\exists$ maximal ideal m s.t. $M_m \neq 0$.

Since $M \neq 0$, pick $x \in M$, $x \neq 0$.

$\text{ann}(x) = \{a \in A \mid ax = 0\}$ is proper ideal.

$\text{ann}(x) \neq A \Rightarrow \exists$ maximal ideal m s.t. $\text{ann}(x) \subseteq m$.

Claim: $\frac{x}{1} \neq 0$ in M_m.

Otherwise, $\exists\ a \notin m$ s.t. $ax = 0 \Rightarrow a \in \text{ann}(x) \rightarrow\leftarrow$.

So $M_m \neq 0$. $\square$

Note. For any prime p, A_p is a local ring.

The maximal ideal is extension pA_p.

If $\frac{a}{s} \notin A_p$, we may assume $a \notin p$.

Then $\frac{s}{a} = \left(\frac{a}{s}\right)^{-1}$, so $\frac{a}{s}$ is a unit.

Also, pA_p is not unit ideal: otherwise $\frac{1}{1} = \frac{a}{s}$ for some $a \in p$. $\Rightarrow \exists\ x \notin p$ s.t. $x(s-a) = 0$.

$\Rightarrow x(s-a) \in p. \Rightarrow s - a \in p$. Since $a \in p \Rightarrow s \in p. \rightarrow\leftarrow$.

Cor. Let $f: M \to N$ be A-module homomorphism. TFAE:

1. f is injective.
2. $f_p: M_p \to N_p$ is injective for all primes p.
3. $f_m: M_m \to N_m$ is injective for all maximal ideals m.

Same result if "injective" replaced by "surjective" or "isomorphism".

Pf. Consider exact sequence $0 \to \ker f \to M \xrightarrow{f} N$.

$$
\begin{aligned}
f \text{ is injective} &\iff \ker f = 0 \\
&\iff (\ker f)_p = 0 \text{ for all primes } p \\
&\iff (\ker f)_m = 0 \text{ for all maximal ideals } m.
\end{aligned}
$$

By exactness of localization, $(\ker f)_p = \ker(f_p)$ for all primes p.

$\implies$ ① $\iff$ ② $\iff$ ③.

To do surjective, consider exact sequence $M \xrightarrow{f} N \to \operatorname{coker} f \to 0$.

Finally, isomorphism = injective + surjective. $\square$

Prop. TFAE: ① M is a flat A-module.

② M_p is a flat A_p-module for all primes p.

③ M_m is a flat A_m-module for all maximal ideals m.

Pf. ① $\Rightarrow$ ②: Let $N_1 \xrightarrow{f} N_2$ be injective map of A_p-modules. Using canonical map $A \to A_p$, f is also an injective map of A-modules.

Since M is flat, then $M \otimes_A N_1 \xrightarrow{1 \otimes f} M \otimes_A N_2$ is injective.

Localize at p: $(M \otimes_A N_1)_p \xrightarrow{(1 \otimes f)_p} (M \otimes_A N_2)_p$ is injective.

From previous lecture:

$$\begin{array}{ccc} (M \otimes_A N_1)_p & \longrightarrow & (M \otimes_A N_2)_p \\ \wr\| & & \wr\| \\ M_p \otimes_{A_p} (N_1)_p & \xrightarrow{1 \otimes f_p} & M_p \otimes_{A_p} (N_2)_p \end{array}$$

Also, $(N_i)_p = N_i$, so tensoring w/ M_p preserved injectivity.

$\Rightarrow$ M_p is flat A_p-module.

(2) $\Rightarrow$ (3) clear.

(3) $\Rightarrow$ (1) Suppose M is not flat. Then $\exists$ injective map $N_1 \xrightarrow{f} N_2$ of A-modules s.t. $M \otimes_A N_1 \longrightarrow M \otimes_A N_2$ is not injective. Let K be its kernel, so $K \neq 0$, $\exists$ maximal ideal m s.t. $K_m \neq 0$. But we can identify

$$K_m = \ker\left((M \otimes_A N_1)_m \xrightarrow{(1\otimes f)_m} (M \otimes_A N_2)_m \right)$$

$$= \ker\left(M_m \otimes_{A_m} (N_1)_m \xrightarrow{1 \otimes f_m} M_m \otimes_{A_m} (N_2)_m \right)$$

However, $f_m : (N_1)_m \to (N_2)_m$ is injective.

$\Rightarrow$ M_m is not a flat A_m-module. $\square$

Extended/contracted ideals wrt $A \to S^{-1}A$.

Given ideal $I \subset A$, have $S^{-1}I = (S^{-1}A)I$

Prop. Every ideal $J \subset S^{-1}A$ is equal to $S^{-1}I$ for some $I \subset A$; can take I = preimage of J along $A \to S^{-1}A$.

Pf. $S^{-1}I \subseteq J$ by definition.

If $\frac{a}{s} \in J$, then $\frac{a}{1} \in J$, so $a \in I$, so $\frac{a}{s} \in S^{-1}I$. $\square$

Prop. Have bijection

$$\{p \in \operatorname{Spec} A \mid p \cap S = \emptyset\} \leftrightarrow \operatorname{Spec}(S^{-1}A).$$

$$p \longrightarrow S^{-1}p.$$

- In particular, for any $f \in A$, $\operatorname{Spec}(A_f)$ is identified w/ open subset $\operatorname{Spec} A \setminus V(f)$.
- Also, for any prime q, $\operatorname{Spec}(A_q)$ is in bijection w/ primes of A contained in q.

Pf. Pick $p \in \operatorname{Spec} A$ s.t. $p \cap S = \emptyset$. Let $\bar{S}$ = image of S under $A \to A/p$.

Then $\bar{S} \not\ni 0$.

We also have $\underbrace{\bar{S}^{-1}(A/p)}_{\text{contained in fraction field of } A/p.} = S^{-1}(A/p)$

Also, $\bar{S}^{-1}(A/p) \cong S^{-1}A/S^{-1}p \Rightarrow S^{-1}p$ is prime in $S^{-1}A$.

We have $\operatorname{Spec}(S^{-1}A) \longrightarrow \{p \in \operatorname{Spec} A \mid p \cap S = \emptyset\}$

$q \longrightarrow$ preimage of q under $A \to S^{-1}A$.

First, consider $g: A/p \to \bar{S}^{-1}(A/p)$. (for $p \cap S = \emptyset$)

which is injective

$$\rightsquigarrow \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & S^{-1}A \arrow[d, "\beta"] \\ A/p \arrow[r, "g"] & \bar{S}^{-1}(A/p) = S^{-1}A/S^{-1}p \end{tikzcd}$$

Then: $f^{-1}(\beta^{-1}(0)) = f^{-1}(S^{-1}p)$

$\parallel$

$\alpha^{-1}(g^{-1}(0)) = \alpha^{-1}(0) = p.$

Second, if $q \in \operatorname{Spec}(S^{-1}A)$, then $S^{-1}p = q$ if p is preimage of q under $A \to S^{-1}A$. $\square$

Prop. S^{-1} commutes w/ finite sums, intersections, products of ideals.

Pf. Let $I, J \subset A$ ideals. Image of $S^{-1}(IJ)$ in $S^{-1}A$ is contained in $(S^{-1}I)(S^{-1}J)$ and is closed under addition, so suffices to show $\frac{x}{s} \cdot \frac{y}{t}$ is in image of $S^{-1}(IJ)$ if $x \in I$, $y \in J$, $s, t \in S$.

But $\frac{x}{s} \cdot \frac{y}{t} = \frac{xy}{st}$ and $xy \in IJ$. $\square$

Prop. S^{-1} commutes w/ taking radicals, i.e., $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$. In particular, if A is reduced, then so is $S^{-1}A$.

Pf. Pick $x \in \sqrt{I}$. Then $x^n \in I$ for some $n \geq 1$. If $s \in S$, then $\left(\frac{x}{s}\right)^n \in S^{-1}I$, so $\frac{x}{s} \in \sqrt{S^{-1}I}$

$\Rightarrow S^{-1}\sqrt{I} \subseteq \sqrt{S^{-1}I}$.

Conversely, suppose $\frac{x}{s} \in \sqrt{S^{-1}I}$. So $\exists n \geq 1$ s.t. $\left(\frac{x}{s}\right)^n \in S^{-1}I$.

We can write $\left(\frac{x}{s}\right)^n = \frac{y}{t}$ for some $y \in I$, $t \in S$.

$\Rightarrow \exists u \in S$ s.t. $u(x^n t - s^n y) = 0$.

Note: $us^n y \in I$, so $ux^n t \in I$. $\Rightarrow u^n x^n t^n \in I$

$\Rightarrow uxt \in \sqrt{I}$.

Finally, $\frac{x}{s} = \frac{uxt}{uts} \in S^{-1}\sqrt{I}$. $\square$