# Localization - Modules

Let $A$ = ring, $S \subset A$ multiplicative subset, $M$ = $A$-module.

Define $\sim$ on $M \times S$ by

$(m,s) \sim (n,t)$ if $\exists\, x \in S$ s.t. $x(tm - sn) = 0$.

$\sim$ is equivalence relation.

Define $S^{-1}M := (M \times S)/\sim$. This is module over $S^{-1}A$:

- $(m,s) + (m',s') = (s'm + sm', ss')$.
- For $(a,t) \in S^{-1}A$, $(a,t)\cdot(m,s) = (am, st)$.

Additive unit in $S^{-1}M$ is $(0,1)$.

Given $f: M \to N$ is homomorphism of $A$-modules, get

$S^{-1}(f): S^{-1}M \to S^{-1}N$ via $S^{-1}(f)\left(\frac{m}{s}\right) = \frac{f(m)}{s}$.

For any $g: N \to P$, have $S^{-1}(g \circ f) = S^{-1}(g) \circ S^{-1}(f)$

i.e., $S^{-1}$ is a functor from $A$-modules to $S^{-1}A$-modules.

Ex. If $S = A \setminus p$, $p$ prime, then $M_p$ is written in place of $S^{-1}M$. Localization of $M$ at $p$.

Prop. $S^{-1}$ is exact, i.e., if $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ is exact, then so is $S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3$.

Pf. Note: $S^{-1}(g) \circ S^{-1}(f) = S^{-1}(g \circ f) = S^{-1}(0) = 0$.

$\Rightarrow$ image $S^{-1}f \subseteq \ker S^{-1}(g)$.

Need equality.

Pick $\frac{m}{s} \in \ker S^{-1}g$. Then $0 = (S^{-1}g)(\frac{m}{s}) = \frac{g(m)}{s}$.

Then $\exists\, x \in S$ s.t. $0 = x\, g(m) = g(xm)$

$\Rightarrow$ $xm \in \ker g$.

$\Rightarrow$ $\exists\, m' \in M$, s.t. $f(m') = xm$,

$$(S^{-1}f)\left(\frac{m'}{xs}\right) = \frac{f(m')}{xs} = \frac{xm}{xs} = \frac{m}{s}.$$

$\Rightarrow$ $\ker S^{-1}(g) = \text{image } S^{-1}f$. $\square$

Special case: $0 \to M \to N$ (i.e., $M$ identified w/ submodule of $N$)

Then $S^{-1}M$ is identified w/ submodule of $S^{-1}N$.

---

Cor. Let $M$ be an $A$-module, $N, P$ submodules of $M$.

(1) $S^{-1}(N+P) = S^{-1}N + S^{-1}P$

(2) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.

(3) $S^{-1}(M/N) \cong S^{-1}M / S^{-1}N$ as $S^{-1}A$-modules

Pf (1) Given $x \in N$, $y \in P$, $s \in S$. Then $\frac{x+y}{s} = \frac{x}{s} + \frac{y}{s}$,

so image of $S^{-1}(N+P)$ in $S^{-1}M$ is contained in $S^{-1}N + S^{-1}P$.

Conversely, given $\frac{x}{s} \in S^{-1}N$, $\frac{y}{t} \in S^{-1}P$, we have

$$\frac{x}{s} + \frac{y}{t} = \frac{tx + sy}{st} \quad \text{and} \quad tx + sy \in N + P.$$

(2) Image of $S^{-1}(N \cap P)$ in $S^{-1}M$ is contained in $S^{-1}N \cap S^{-1}P$.

Pick element in $S^{-1}N \cap S^{-1}P$. Write it as either

$\frac{x}{s}$ or $\frac{y}{t}$ where $x \in N$, $y \in P$, $s,t \in S$.

$\Rightarrow \exists\, u \in S$ s.t. $u(tx - sy) = 0$.

Set $z = utx$. Then $z \in N$.

But also, $z = usy \Rightarrow z \in P \Rightarrow z \in N \cap P$.

Note: $\frac{z}{uts} = \frac{utx}{uts} = \frac{x}{s}$, so $\frac{z}{uts} \to \frac{x}{s}$ under the map $S^{-1}(N \cap P) \to S^{-1}M$.

(3) Apply $S^{-1}$ to short exact sequence

$$0 \to N \to M \to M/N \to 0$$

to get

$$0 \to S^{-1}N \to S^{-1}M \to S^{-1}(M/N) \to 0$$

$$\Rightarrow \quad S^{-1}(M/N) \cong S^{-1}M/S^{-1}N. \qquad \square$$

---

Base change gives another functor from $A$-modules to $S^{-1}A$-modules.

Given $A$-module $M$, get $S^{-1}A \otimes_A M$

Prop. The map $S^{-1}A \otimes_A M \longrightarrow S^{-1}M$ given by

$$\sum_i \left(\frac{a_i}{s_i}\right) \otimes m_i \to \sum_i \frac{a_i m_i}{s_i}$$

is an isomorphism of $S^{-1}A$-modules.

Rmk. (Naturality) given $\varphi: M \to N$, get commutative square

$$\begin{array}{ccc} S^{-1}A \otimes_A M & \longrightarrow & S^{-1}M \\ {\scriptstyle id \otimes \varphi}\downarrow & & \downarrow{\scriptstyle S^{-1}(\varphi)} \\ S^{-1}A \otimes_A N & \longrightarrow & S^{-1}N \end{array}$$

Pf. Well-defined: this comes from bilinear function

$$S^{-1}A \times M \longrightarrow S^{-1}M$$
$$\left(\left(\frac{a}{s}\right), m\right) \longrightarrow \frac{am}{s}.$$

Surjectivity: clear. $\frac{m}{s}$ is in image of $\left(\frac{1}{s}\right)\otimes m$.

Injectivity: Suppose we have $\sum_{i=1}^{n}\left(\frac{a_i}{s_i}\right)\otimes m_i \to 0$

i.e., $$\sum_{i=1}^{n}\frac{a_i m_i}{s_i} = 0$$

Define $s = s_1 s_2 \cdots s_n$, $s_j' = s_1 s_2 \cdots \hat{s_j} \cdots s_n$

$$\Rightarrow 0 = \sum_{i=1}^{n}\frac{a_i m_i}{s_i} = \sum_{i=1}^{n}\frac{a_i m_i s_i'}{s} = \frac{\sum_{i=1}^{n} a_i m_i s_i'}{s}.$$

$$\Rightarrow \exists x \in S \text{ s.t. } x\sum_{i=1}^{n} a_i m_i s_i' = 0$$

Now, $$\sum_{i=1}^{n}\left(\frac{a_i}{s_i}\right)\otimes m_i = \sum_{i=1}^{n}\left(\frac{a_i s_i' x}{s x}\right)\otimes m_i = \sum_{i=1}^{n}\left(\frac{1}{sx}\right)\otimes a_i s_i' x m_i$$

$$= \frac{1}{sx}\otimes\sum_{i=1}^{n} a_i s_i' x m_i = \frac{1}{sx}\otimes 0 = 0. \qquad \square$$

Cor. $S^{-1}A$ is a flat $A$-module.

---

Prop. $M, N$ = $A$-modules. The map

$$S^{-1}M \underset{S^{-1}A}{\otimes} S^{-1}N \longrightarrow S^{-1}(M \underset{A}{\otimes} N)$$

$$\sum_i\left(\frac{m_i}{s_i}\right)\otimes\left(\frac{n_i}{t_i}\right) \longmapsto \sum_i \frac{m_i\otimes n_i}{s_i t_i}$$ is isomorphism.

Pf. Well-defined: comes from bilinear map

$$S^{-1}M \times S^{-1}N \longrightarrow S^{-1}(M \otimes_A N)$$

$$\left( \left(\frac{m_i}{s_i}\right), \left(\frac{n_i}{t_i}\right) \right) \longrightarrow \frac{m_i \otimes n_i}{s_i t_i},$$

Inverse of this map: $S^{-1}(M \otimes_A N) \longrightarrow S^{-1}M \otimes_{S^{-1}A} S^{-1}N$

$$\frac{\sum m_i \otimes n_i}{s} \longmapsto \sum \left(\frac{m_i}{1}\right) \otimes \left(\frac{n_i}{s}\right) \qquad \square$$