Associated primes

$I \subset A$ ideal,

A prime $p \subset A$ is associated to I if $\exists x \in A$

s.t. $p = (I : x) = \{a \in A \mid ax \in I\}$

$\Leftrightarrow$ $p = \mathrm{Ann}_{A/I}(x)$

$\Leftrightarrow$ $A/p \to A/I$ given by $a + p \to ax + I$ is well-defined and injective

$AP_A(I)$ = set of associated primes of I.

If $I = (y)$, y is nonzerodivisor, any associated prime of I is called associated to a nonzerodivisor.

For $I = 0$, an associated prime to 0 is called associated prime of A.

Prop. Consider poset of ideals $\{\mathrm{Ann}(x) \mid x \neq 0\}$.

Any maximal element of this poset is prime and in particular, is associated prime of A.

Pf. let $I = \mathrm{Ann}(x)$ be maximal element, w/ $x \neq 0$.

$\Rightarrow$ $1 \notin I \Rightarrow I \neq A$.

Suppose $ab \in I$, $a \notin I$. Then $ax \neq 0$, and $I \subseteq \mathrm{Ann}(ax)$

$\Rightarrow$ $I = \mathrm{Ann}(ax)$.

Since $ab \in I$, $abx = 0 \Rightarrow b \in \mathrm{Ann}(ax) = I$.

$\Rightarrow$ I prime. $\square$

If A is noethenian, associated primes exist.
(and $A \neq 0$)

Cor. Let A = noethenian ring. Pick $x \in A$, $x \neq 0$.
Then $\exists$ associated prime $\mathfrak{p}$ s.t. x is nonzero in $A_{\mathfrak{p}}$.

pf. Consider set $\{ \mathrm{Ann}(y) \mid y \neq 0 \ \& \ \mathrm{Ann}(x) \subseteq \mathrm{Ann}(y) \}$

A noeth $\Rightarrow$ it has maximal element $\mathfrak{p}$.

Then x is nonzero in $A_{\mathfrak{p}}$. $\square$

Given ring A, let S = set of nonzerodivisors.
$A[S^{-1}]$ = total ring of fractions of A.

Prop. A = noeth. ring. B = total ring of fractions.
Pick $x \in B$. Then, $x \in A \iff$ image of x in $B_{\mathfrak{p}}$ belongs to $A_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ associated to a nonzerodivisor.

Pf. Write $x = \frac{a}{u}$, $a \in A$, $u \in A$, u nonzerodivisor.

Suppose $x \notin A$. Then $a \notin (u)$, so a is nonzero in $A/(u)$.
Then $\exists$ associated prime $\mathfrak{p}' \in AP_{A/u}(0)$ s.t. image of a in $(A/u)_{\mathfrak{p}'}$ is nonzero.
Let $\mathfrak{p}$ be inverse image of $\mathfrak{p}'$ in A. Then
$\mathfrak{p} \in AP_A((u))$ & $(A/u)_{\mathfrak{p}'} = A_{\mathfrak{p}}/(u)_{\mathfrak{p}}$

$\Rightarrow a \notin (u)_p. \Rightarrow a/u \notin A_p.$ $\square$

Serre conditions.

Let $d \geq 0$ integer. A = ring.

① A satisfies (R_d) if, for every prime $p \subset A$ of height $\leq d$ (i.e., $\dim A_p \leq d$) A_p is a regular local ring.

② (a) A satisfies (S_1) if every prime associated to 0 has height 0.

(b) A satisfies (S_2) if satisfies (S_1) & every prime associated to a nonzerodivisor has height 1.

Reduced rings

(A is reduced if it has no nonzero nilpotent elements).

Prop. Let A = 0-dim noeth. local ring, m = maximal ideal. TFAE:

① A is a regular local ring (i.e, $m = 0$)

② A is a field.

③ A is reduced.

Pf. ① $\Leftrightarrow$ ② ✓ ② $\Rightarrow$ ③ ✓

③ $\Rightarrow$ ① : m = set of nilpotents, so $m = 0$. $\square$

Thm. Let A = noeth. ring. Then A is reduced $\Longleftrightarrow$ A satisfies (R_0) & (S_1).

pf. Suppose A is reduced. Let $p \subset A$ be prime of height 0. Then A_p has dim 0 and also reduced:

suppose $\frac{x}{s} \in A_p$ is nilpotent, so $\frac{x^n}{s^n} = 0$

$\Rightarrow t \notin p$ s.t. $tx^n = 0 \Rightarrow (tx)^n = 0 \Rightarrow tx = 0$

$\Rightarrow \frac{x}{s} = 0$ in A_p.

$\Rightarrow A_p$ is regular local ring,

so A satisfies (R_0).

Let $p \in AP_A(0)$, so $p = \mathrm{Ann}(x)$ where $x \neq 0$.

Suppose height$(p) > 0$. Then $\exists$ prime $q \subsetneq p$. Pick $y \in p \setminus q$. Then $yx = 0 \in q \Rightarrow x \in q$.

$\Rightarrow x \in p \Rightarrow x^2 = 0 \to\!\leftarrow$ since A is reduced.

$\Rightarrow A$ satisfies (S_1). //

Now suppose A satisfies (R_0) & (S_1).

Pick $x \in A$ nilpotent. Suppose $x \neq 0$.

$\exists$ $p \in AP_A(0)$ s.t. x is nonzero in A_p.

By (S_1), height$(p) = 0$, so $\dim(A_p) = 0$.

By (R_0), $A_\mathfrak{p}$ is regular local ring, hence reduced

But, x is nonzero and nilpotent in $A_\mathfrak{p}$ $\rightarrow\leftarrow$.

$\Longrightarrow$ A is reduced. $\square$

Recall: topological space is irreducible if it cannot be expressed as union of 2 closed proper subsets.

Prop. A = ring. Then A is a domain $\Longleftrightarrow$ A is reduced & $\operatorname{Spec}(A)$ is irreducible.