Completion and associated graded

Thm (Krull intersection Theorem). A = noeth. ring, I = ideal, M = f.g. A-module, $\hat{M}$ = I-adic completion of M. The kernel of $M \longrightarrow \hat{M}$ is

$$\bigcap_{i=1}^{\infty} I^i M = \left\{ x \in M \;\middle|\; \begin{array}{l} ax = 0 \text{ for some } a \in A \text{ s.t.} \\ a - 1 \in I \end{array} \right\}$$

Pf. Define $E = \bigcap_{i=1}^{\infty} I^i M$

By Artin-Rees, $\exists\, k$ s.t. $\underbrace{E \cap I^{k+1} M}_{E} = I(\underbrace{E \cap I^k M}_{IE})$

$\Rightarrow E = IE$. By Nakayama's lemma, $\exists\, a \in A$ s.t. $a - 1 \in I$ and $ax = 0$ for all $x \in E$.

Suppose $x \in M$ and $\exists\, a \in A$ s.t. $ax = 0$ & $a - 1 \in I$.

$\Rightarrow (a-1)x = -x \Rightarrow (a-1)^i x = (-1)^i x$

$\Rightarrow x \in I^i M \;\; \forall i \Rightarrow x \in E$ $\square$

If $\ker(M \to \hat{M}) = 0$, then M is <u>separated with respect to I</u>.

Cor. A = noeth. local ring w/ maximal ideal m. Any f.g module M is separated w.r.t. m.

Pf. If $a - 1 \in m$, then $a \notin m \Rightarrow a$ is unit.

$$\Rightarrow \bigcap_{i=1}^{\infty} m^i M = 0 .$$ $\square$

Cor. If A is noeth. domain, then A is separated w.r.t. any proper ideal.

Prop. A = ring, I = ideal, M = A-module. $\mathcal{F} = (M_n)$ = I-filtration of M

Suppose that: ① A is I-adically complete.

② $\bigcap_n M_n = 0$

③ $gr_{\mathcal{F}}(M)$ is a f.g. $gr_I(A)$-module.

Then: M is a f.g. A-module.

Pf. A is I-adically complete, $A \to \hat{A}$ is isom, so $\bigcap_n I^n = 0$. So for any nonzero $a \in A$, $\exists$ largest integer $n \geq 0$ s.t. $a \in I^n$ $[I^0 = A]$

Define initial term $\mathrm{init}(a) \in I^n / I^{n+1} = (gr_I(A))_n$

For nonzero $m \in M$, $\exists$ largest integer $n \geq 0$ s.t. $m \in M_n$

define initial term $\mathrm{init}(m) \in M_n / M_{n+1} = (gr_{\mathcal{F}}(M))_n$

Claim: If $m_1, \dots, m_r \in M$ are elements s.t. $\mathrm{init}(m_1), \dots \mathrm{init}(m_r)$ generated $gr_{\mathcal{F}}(M)$ as $gr_I(A)$-module, Then $m_1, \dots m_r$ generate M as A-module.

Pf of Claim: Pick nonzero $x_0 \in M$. We have expression

$\mathrm{init}(x_0) = c_{0,1}\,\mathrm{init}(m_1) + \dots + c_{0,r}\,\mathrm{init}(m_r)$

where $c_{0,i}$ is homogeneous of degree $\mathrm{init}(x_0) - \mathrm{init}(m_i)$

[$c_{0,i} = 0$ if $\mathrm{init}(x_0) < \mathrm{init}(m_i)$]

$\exists\, C_{0,i} \in A$ s.t. $c_{0,i} = \mathrm{init}(C_{0,i})$.

Define $x_1 = x_0 - (C_{0,1} m_1 + \cdots + C_{0,r} m_r)$

$\Rightarrow$ $\deg \mathrm{init}(x_1) > \deg \mathrm{init}(x_0)$ if $x_1 \neq 0$.

[if $x_1 = 0$, then x_0 is linear comb of $m_1, \ldots m_r$]

Repeating above we get sequence $x_0, x_1, x_2, \ldots$ s.t.

- $\deg \mathrm{init}(x_{i+1}) > \deg \mathrm{init}(x_i)$ for all i,
- $x_{i+1} = x_i - (C_{i,1} m_1 + \cdots + C_{i,r} m_r)$

for $C_{i,j} \in A$ s.t. $\deg(\mathrm{init}(C_{ij})) = \deg \mathrm{init}(x_i) - \deg \mathrm{init}(m_j)$

if $C_{ij} \neq 0$.

$\Rightarrow$ for fixed j, $\deg \mathrm{init}\, C_{ij}$ is increasing w/ i for the nonzero C_{ij}

$\Rightarrow$ $C_j := \sum_i C_{ij}$ is well-defined element of A [since A complete]

$x_0 - (C_1 m_1 + \cdots + C_r m_r)$ is 0 modulo M_n for all $n \gg 0$. $\Rightarrow$ must have $x_0 = C_1 m_1 + \cdots + C_r m_r$ since $\bigcap_n M_n = 0$. $\square$

Thm. If A noeth, $I \subset A$ ideal, then $\hat{A}$ is noeth.

Pf. Let $J \subset \hat{A}$ be ideal.

Then $gr_{\hat{I}}(J)$ is an ideal in $gr_{\hat{I}}(\hat{A})$

Next, $gr_{\hat{I}}(\hat{A}) \cong gr_I(A)$ b/c $\hat{I}^n/\hat{I}^{n+1} \cong I^n/I^{n+1}$

noeth

$\Rightarrow$ $gr_{\hat{I}}(J)$ is finitely generated

Next, $\bigcap_n \hat{I}^n J \subseteq \bigcap_n \hat{I}^n = 0$

$\Rightarrow$ J is f.g. $\Rightarrow$ $\hat{A}$ is noeth. $\square$

Cor. If A = noeth, then $A[[x_1, \dots, x_n]]$ is noeth. for any n.

Prop. A = noeth. local ring w/ maximal ideal m. Let $\hat{A}$ = m-adic completion. Then $\dim A = \dim \hat{A}$

Furthermore, A is regular local ring $\Leftrightarrow$ $\hat{A}$ is regular local ring.

Pf. We know $gr_m(A) \cong gr_{\hat{m}}(\hat{A})$, and this determines dimension. $\Rightarrow$ $\dim A = \dim \hat{A}$.

Second, $m/m^2 \cong \hat{m}/\hat{m}^2$, and $k = A/m \cong \hat{A}/\hat{m}$ $\square$