Completion of modules

A = ring, I = ideal, M = A-module

I-adic filtration of M: $M_i = I^i M$

$$M = M_0 \supset M_1 \supset \cdots$$

I-adic topology on M is corresponding Krull topology

I-adic completion of M is $\hat{M} = \varprojlim_i M/I^i M$

$$M \longrightarrow \hat{M}$$
$$m \longrightarrow (m + I^i M)_i$$

M is complete (with respect to I) if this map is bijection

$\hat{M}$ has structure of $\hat{A}$-module:

for $(a_i) \in \hat{A}$, $(m_i) \in \hat{M}$, $(a_i)\cdot(m_i) = (a_i m_i) \in \hat{M}$.

Given A-linear $f: M \to N$, $\hat{f}: \hat{M} \to \hat{N}$ is $\hat{A}$-linear.

Rmk: completion is a functor from category of A-modules to category of $\hat{A}$-modules.

Rmk. Suppose A is complete wrt I. Let $a_0, a_1, \ldots \in A$ nonzero.

For each i, $\exists$ largest integer $n(i)$ s.t. $a_i \in I^{n(i)}$.

Assume for all n, $\{ i \mid n(i) \leq n \}$ is finite.

Then infinite sum $\sum_{i=0}^{\infty} a_i$ makes sense: as sequence, the n^{th} term is sum of $a_i + I^n$ (only has finitely many nonzero elements)

Reminder: filtration $M_0 \supseteq M_1 \supseteq \cdots$ is I-stable if

$$I M_i = M_{i+1} \text{ for } i \gg 0.$$

Prop. The Krull topology given by I-stable filtration is the I-adic topology. In particular, $\varprojlim_i M/M_i \cong \hat{M}$.

Pf. $\exists\, n_0$ s.t. $M_{i+n_0} \subseteq I^i M$ & $I^{i+n_0} M \subseteq M_i$ for all i.

$\Rightarrow$ $I^i M$ is a union of cosets of M_{i+n_0}

M_i is a union of cosets of $I^{i+n_0} M$

$\Rightarrow$ both give same Krull topology. $\square$

Prop. Let A be noeth, given short exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

of finitely generated A-modules. Then completion is exact:

$$0 \to \hat{M}' \xrightarrow{\hat{f}} \hat{M} \xrightarrow{\hat{g}} \hat{M}'' \to 0 \quad \text{is exact sequence}$$

of $\hat{A}$-modules.

Pf. Artin-Rees lemma: $M' \cap I^i M$ is I-stable filtration of M'.

$\Rightarrow$ subspace topology on M' given by I-adic topology on M is I-adic topology on M'.

Also: $g(I^i M) = I^i M''$, so induced filtration is also I-adic filtration.

$\Rightarrow$ get $0 \to \hat{M}' \xrightarrow{\hat{f}} \hat{M} \xrightarrow{\hat{g}} \hat{M}'' \to 0$ exact sequence of abelian groups.

But, $\hat{f}, \hat{g}$ are $\hat{A}$-linear, exactness only uses abelian group structure. $\square$

Name the map $\varphi_M : M \longrightarrow \hat{M}$

$m \longrightarrow (m + I^i M)_i$

$\varphi_A : A \to \hat{A}$ gives $\hat{A}$ the structure of A-module.

Define $\Phi_M : \hat{A} \otimes_A M \longrightarrow \hat{M}$ homomorphism of $\hat{A}$-modules.

$$\sum \alpha_i \otimes m_i \longrightarrow \sum \alpha_i \varphi_M(m_i)$$

Given $f : M \to N$ A-linear, we have commutative diagram.

$$\begin{tikzcd} \hat{A} \otimes_A M \arrow[r, "\Phi_M"] \arrow[d, "1\otimes f"'] & \hat{M} \arrow[d, "\hat{f}"] \\ \hat{A} \otimes_A N \arrow[r, "\Phi_N"] & \hat{N} \end{tikzcd}$$

Rmk. This says that Φ is a "natural transformation" between "tensor w/ $\hat{A}$" and "completion"

Thm. If M is finitely generated, Φ_M is surjective.

If A is noetherain, then Φ_M is isomorphism for any f.g M.

$\Rightarrow$ If A is noetherain, $\hat{A}$ is a flat A-module.

Pf. M f.g. $\Rightarrow$ $\exists$ short exact sequence

$$0 \to K \to A^n \xrightarrow{f} M \to 0$$

$\rightsquigarrow$ commutative diagram

$$\begin{tikzcd} \hat{A} \otimes_A A^n \arrow[r, "\Phi_{A^n}"] \arrow[d, "1\otimes f"'] & \widehat{A^n} \arrow[d, "\hat{f}"] \\ \hat{A} \otimes_A M \arrow[r, "\Phi_M"] & \hat{M} \end{tikzcd}$$

$1\otimes f$ is surjective by right-exactness of tensor product.

Φ_{A^n} is isomorphism by direct calculation

$\hat{f}$ is surjective by earlier result on inverse limits

$\Longrightarrow$ Φ_M is also surjective.

Now suppose A is noetherian. $\Rightarrow$ K is f.g. Have diagram

$$\begin{tikzcd}
& \hat{A} \otimes_A K \arrow[r] \arrow[d, "\Phi_K"'] & \hat{A} \otimes_A A^n \arrow[r] \arrow[d, "\Phi_{A^n}"'] & \hat{A} \otimes_A M \arrow[r] \arrow[d, "\Phi_M"] & 0 \\
0 \arrow[r] & \hat{K} \arrow[r] & \hat{A}^n \arrow[r] & \hat{M} \arrow[r] & 0
\end{tikzcd}$$

commutes by naturality of Φ.

Snake lemma $\Rightarrow$ exact sequence

$$\underbrace{\cancel{\ker \Phi_{A^n}}}_{0} \to \ker \Phi_M \to \overbrace{\cancel{\operatorname{coker} \Phi_K}}^{0}$$

since Φ_{A^n} is isom.

K f.g. $\Rightarrow$ Φ_K surjective

$\Rightarrow$ $\ker \Phi_M = 0$ $\Rightarrow$ Φ_M is isom. $\square$

<u>Rmk.</u> Let A be noeth. When restricting domain to finitely generated A-modules Φ gives isomorphism between functors "tensor w/ $\hat{A}$" and "completion"

<u>Note</u>: They need not be isomorphic in general.

in fact, "completion" need not be exact for non f.g. modules.

However "tensor w/ $\hat{A}$" is exact for non f.g.-modules