Localization

A = ring

$S \subset A$ is a <u>multiplicative subset</u> if $1 \in S$, and

$x, y \in S \Rightarrow xy \in S$

Consider $A \times S$, put relation $\sim$ as follows:

$(a,s) \sim (b,t)$ if $\exists x \in S$ s.t. $x(at - bs) = 0$.

- symmetry, reflexive properties are clear
- transitive: suppose $(b,t) \sim (c,u)$, i.e., $\exists y \in S$ s.t.

$$y(bu - ct) = 0.$$

Then: $0 = (at - bs)xuy + (bu - ct)ysx$

$= atxuy - bs\cancel{x}uy + b\cancel{u}ysx - ctysx$

$= (au - cs)(txy)^{\in S} \Rightarrow (a,s) \sim (c,u)$

Define $S^{-1}A := (A \times S)/\sim$ (ring of fractions of A w.r.t. S)

Intuition: (a,s) is fraction $\frac{a}{s}$ (sometimes a/s)

$S^{-1}A$ has ring structure:

- $(a,s) + (b,t) = (at + bs, st)$
- $(a,s)(b,t) = (ab, st)$

<u>check</u>: this is independent of equivalence representative

The additive unit is $(0,1)$

Multiplicative unit is $(1,1)$.

<u>Examples</u>: ① If $p \subset A$ is prime ideal, then $S = A \setminus p$ is multiplicative set. Write A_p in place of $S^{-1}A$.

This is localization of A at p.

(1b) Special case: A is domain, $p = (0)$.
Then $A_{(0)}$ is field of fractions of A.

If $A = \mathbb{Z}$, $A_{(0)} = \mathbb{Q}$.

(2) Given $f \in A$, $S = \{1, f, f^2, f^3, \dots\}$ is multiplicative.
Write A_f or $A[1/f]$ instead of $S^{-1}A$.

There is canonical homomorphism:

$$f : A \longrightarrow S^{-1}A$$
$$f(a) = (a, 1)$$

[Note: if $\exists\, x \in S$ s.t. $xa = 0$, then $(a,1) = 0$]

Prop. If $g : A \to B$ is a ring homomorphism s.t. $g(s)$ is invertible for all $s \in S$, then $\exists$ unique homomorphism $h : S^{-1}A \to B$ s.t. $g = h \circ f$.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \\ S^{-1}A \arrow[ur, dotted, "h"'] & \end{tikzcd}$$

Pf. Define $h\left(\frac{a}{s}\right) = g(a) g(s)^{-1}$.

If $\frac{a}{s} = \frac{b}{t}$, then $\exists\, x \in S$ s.t. $x(at - bs) = 0$.

So: $g(x)\big(g(a)g(t) - g(b)g(s)\big) = 0$.

Multiply by $g(tsx)^{-1} = g(t)^{-1} g(s)^{-1} g(x)^{-1}$

$\Rightarrow g(a) g(s)^{-1} - g(b) g(t)^{-1} = 0 \Rightarrow h$ well-defined.

Uniqueness: let h' be any other homomorphism s.t. $g = h' \circ f$

Then $h'\left(\frac{a}{s}\right) = h'\left(\frac{a}{1}\right) h'\left(\frac{1}{s}\right) = g(a)\, g(s)^{-1} = h\left(\frac{a}{s}\right)$.

since $g(a) = h'(f(a)) = h'\left(\frac{a}{1}\right)$ ...

□