Inverse Limits

Setup: $G_0, G_1, G_2 \ldots$ abelian groups
$\theta_n : G_n \to G_{n-1}$ group homomorphism for all $n \geq 1$ } inverse system

Denote it by (G_i, θ_i) or just $\{G_i\}$

The inverse limit is

$$\varprojlim_i G_i = \{(g_0, g_1, \ldots) \in \prod_{i \geq 0} G_i \mid \theta_i(g_i) = g_{i-1} \ \forall i \geq 1\}$$

subgroup of $\prod_{i \geq 0} G_i$

If G_i are rings, θ_i ring homomorphisms, $\varprojlim_i G_i$ is also a ring.

$(G_i, \theta_i), (G_i', \theta_i')$ are inverse systems.

A morphism $f : \{G_i\} \to \{G_i'\}$ is a sequence of homomorphisms
$f_i : G_i \to G_i'$ s.t.

$$\begin{tikzcd} G_i \arrow[r, "f_i"] \arrow[d, "\theta_i"'] & G_i' \arrow[d, "\theta_i'"] \\ G_{i-1} \arrow[r, "f_{i-1}"] & G_{i-1}' \end{tikzcd}$$

commutes for all $i \geq 1$.

f is injective (resp. surjective) if all f_i have that property

For abelian groups, $\{A_i\} \xrightarrow{f} \{B_i\} \xrightarrow{g} \{C_i\}$ is exact if
$A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i$ is exact for all i.

An inverse system (A_i, θ_i) is surjective if θ_i are surjective for all $i \geq 1$.

Prop. Given $0 \to \{A_i\} \to \{B_i\} \to \{C_i\} \to 0$ short exact sequence of inverse systems, get exact sequence $0 \to \varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i$

If $\{A_i\}$ is surjective, get short exact sequence

$$0 \to \varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i \to 0$$

pf. Define $A = \prod_{i\geq 0} A_i$. Define

$$d^A : A \to A \qquad d^A((a_i)_i) = (a_i - \theta_{i+1}(a_{i+1}))_i$$

By definition, $\varprojlim_i A_i = \ker d^A$.

Define B, C, d^B, d^C similarly. We have short exact sequence on direct products

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "d^A"'] & B \arrow[r] \arrow[d, "d^B"'] & C \arrow[r] \arrow[d, "d^C"'] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

SNAKE Lemma : get exact sequence

$$0 \to \varprojlim_i A_i \to \varprojlim_i B_i \to \varprojlim_i C_i \to \operatorname{coker} d^A \to \operatorname{coker} d^B \to \operatorname{coker} d^C \to 0$$

Claim: If $\{A_i\}$ is surjective, then d^A is surjective.

Pick $(a_i)_i \in A$. want $(\alpha_i)_i \in A$ s.t. $d^A(\alpha) = a$.

Construct α_i by induction on i. Set $\alpha_0 = 0$.

Given $\alpha_0, \dots, \alpha_n$ s.t. $\alpha_i - \theta_{i+1}(\alpha_{i+1}) = a_i$ for $i = 0, \dots, n-1$,

let $\alpha_{n+1} \in A_{n+1}$ be any element s.t. (possible since θ_{n+1} surjective).

$$\theta_{n+1}(\alpha_{n+1}) = \alpha_n - a_n \implies \alpha_n - \theta_{n+1}(\alpha_{n+1}) = a_n \qquad \square$$

Important example: P = abelian group.

$P = P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$ decreasing sequence of subgroups.

Set $G_i = P/P_i$, $\theta_i : P/P_i \to P/P_{i-1}$ be natural quotient map. (always surjective)

Set $\hat{P} := \varprojlim_i P/P_i$

There is map $P \to \hat{P}$

$g \to (g + P_i)_i$

If P is ring, P_i are ideals, $\hat{P}$ is also a ring.

EX. Let $\mathbb{k}$ be a ring. $P = \mathbb{k}[x]$

$P_i = (x^i)$. $\hat{P} \cong \mathbb{k}[[x]]$: a power series is

$$f(x) = \sum_{i \geq 0} a_i x^i, \quad a_i \in \mathbb{k}. \iff (a_0 + a_1 x + \cdots + a_{i-1} x^{i-1} \pmod{x^i})$$

image of $P \to \hat{P}$ is power series s.t. $a_i = 0$ for $i \gg 0$.

Ex. $P = \mathbb{Z}$, p = prime, $P_i = (p^i)$, $\hat{P}$ = "ring of p-adic integers" denoted $\mathbb{Z}_p$. By definition, elements of $\mathbb{Z}_p$ are sequences $(a_0, a_1, a_2, \ldots)$ s.t. $a_i \in \mathbb{Z}/p^i$ and $a_i \equiv a_{i+1} \bmod (p^i)$

Use convention that coset representatives for $\mathbb{Z}/n$ are $\{0, \ldots, n-1\}$.

Then there exist integers $b_0, b_1, \ldots$ w/ $0 \leq b_i \leq p-1$ s.t.

$$a_i = b_0 + b_1 p + \cdots + b_{i-1} p^{i-1}.$$

Can use b's to represent elements of $\mathbb{Z}_p$ as infinite sums

$$\sum_{i \geq 0} b_i p^i \quad (0 \leq b_i \leq p-1)$$

Note: $-1 \in \mathbb{Z}_p$ is sum $\sum_{i \geq 0} (p-1) p^i$

Note: $\mathbb{Z}_p$ is local, w/ maximal ideal generated by p.

$\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$ = field of "p-adic numbers"

Cor. Let $0 \to P' \to P \xrightarrow{f} P'' \to 0$ be a short exact sequence of abelian groups. Given $P = P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$

define $P'_i = P' \cap P_i$, $P''_i = f(P_i)$.

Then we get short exact sequence

$$0 \to \hat{P}' \to \hat{P} \to \hat{P}'' \to 0.$$

In particular, take $\Gamma' = \Gamma_n$, get identification of $\hat{\Gamma}_n$ as subgroup of $\hat{\Gamma}$. Get filtration

$$\hat{\Gamma} = \hat{\Gamma}_0 \supseteq \hat{\Gamma}_1 \supseteq \hat{\Gamma}_2 \supseteq \cdots$$

can take inverse limit again: $\hat{\hat{\Gamma}} = \varprojlim_i \hat{\Gamma}/\hat{\Gamma}_i$.

<u>Prop.</u> The natural map $\hat{\Gamma} \to \hat{\hat{\Gamma}}$ is isomorphism.

<u>Pf.</u> For each n, apply corollary above to short exact sequence $0 \to \Gamma_n \to \Gamma \to \Gamma/\Gamma_n \to 0$

Filtration for Γ/Γ_n is $\Gamma_0/\Gamma_n \supseteq \Gamma_1/\Gamma_n \supseteq \Gamma_2/\Gamma_n \supseteq \cdots \supseteq \Gamma_{n-1}/\Gamma_n \supseteq 0 \cdots$

$\Rightarrow$ $(\Gamma/\Gamma_n)_i = (\Gamma/\Gamma_n)/0 = \Gamma/\Gamma_n$ for $i \geq n$

and maps between them are identity.

$\Rightarrow$ the map $\Gamma/\Gamma_n \to \widehat{\Gamma/\Gamma_n}$ is an isomorphism

$$\Rightarrow \hat{\hat{\Gamma}} \to \varprojlim_n \hat{\Gamma}/\hat{\Gamma}_n \cong \varprojlim_n \widehat{\Gamma/\Gamma_n} = \varprojlim_n \Gamma/\Gamma_n = \hat{\Gamma} \qquad \square$$