Transcendental Dimension

Prop. A = d-dimensional noetherian local ring, m = maximal ideal.

$x_1, \dots x_d$ system of parameters, $q = (x_1, \dots, x_d)$

let $f \in A[t_1, \dots t_d]$ homogeneous of degree s s.t.

$f(x_1, \dots x_d) \in q^{s+1}$.

let $\bar{f}$ = reduction of modulo q. $\boxed{\text{Then}}$ $\bar{f}$ is a zerodivisor in

$(A/q)[t_1, \dots, t_d]$.

Pf. Define $\alpha : (A/q)[t_1, \dots t_d] \longrightarrow gr_q(A) = \bigoplus_{i \geq 0} A/q^{i+1}$

$\alpha(t_i) = x_i$ (in q/q^2)

Since f homog of deg s, $\alpha(\bar{f}) \in q^s/q^{s+1}$, so $\bar{f} \in \ker \alpha$.

$\Rightarrow$ have surjection $(A/q)[t_1, \dots t_d]/\bar{f} \rightarrow gr_q(A)$

$\Rightarrow$ For all n, have $\ell\left(\left((A/q)[t_1, \dots t_d]/\bar{f}\right)_n\right) \geq (gr_q(A))_n$

For $n \gg 0$, both sides given by polynomial $(*)$ poly of deg $d-1$

First, $n \to \ell(((A/q)[t_1, \dots t_d])_n)$

is polynomial of deg $d-1$.

Suppose $\bar{f}$ not zerodivisor. Then have exact sequence

$$0 \to (A/q)[t_1, \dots t_d]_{n-s} \xrightarrow{\cdot \bar{f}} (A/q)[t_1, \dots t_d]_n \to \left((A/q)[t_1, \dots t_d]/(\bar{f})\right)_n \to 0$$

$\Rightarrow$ $(*)$ has degree $\leq d-2$ $\rightarrow\leftarrow$

$\Rightarrow$ $\bar{f}$ is a zerodivisor $\square$

Cor. let A = d-dim noetherian local ring, m = maximal ideal. Assume $\exists$ field $\mathbb{k} \subset A$. Then any system of parameters in A is algebraically independent over $\mathbb{k}$.

Pf. Suppose $x_1, .. x_d$ is system of parameters which is algebraically dependent over $\mathbb{k}$. So $\exists f_{\neq 0} \in \mathbb{k}[t_1, .. t_d]$ s.t. $f(x_1, \dots, x_d) = 0$.

Let s = smallest degree of nonzero monomial in f

Let f_s = sum of degree s terms in f; let $g = f - f_s$.

$\Rightarrow g(x_1, .. x_d) \in (x_1, .. x_d)^{s+1}$

$\Rightarrow f_s(x_1, .. x_d) \in (x_1, \dots x_d)^{s+1}$

$\Rightarrow \overline{f_s} \in A/(x_1, .. x_d)[t_1, .. t_d]$ is a zerodivisor.

But, $\mathbb{k} \to A \to A/(x_1, .. x_d)$ is injective, so

$\overline{f_s}$ has coeff. in $\mathbb{k}$ and is "same polynomial" as f_s.

And any polynomial whose coefficients are units cannot be zerodivisor. $\rightarrow\leftarrow$ $\square$

Suppose $\mathbb{k}$ field, A is f.g. $\mathbb{k}$-algebra (which is a domain).

$\dim A = \sup_m \{ \dim A_m \}$

If $\dim A_m = d$, $\exists f_1, .. f_d \in \mathrm{Frac}(A)$ which are alg. ind. over $\mathbb{k}$.

Recall. Given field extension $\mathbb{k} \to E$, a transcendence basis for E over $\mathbb{k}$ is a maximal set of algebraically independent elements

① All transcendence bases have same size. (call it $\underline{\text{transcendence degree}}$)

② If E is f.g. over $\mathbb{k}$ as a field, then transcendence degree is finite.

__Thm__. Let $\mathbb{k}$ = algebraically closed field.

A = f.g. $\mathbb{k}$-algebra which is domain.

__Then__ $\dim A$ = transcendence degree of $\operatorname{Frac}(A) / \mathbb{k}$.

Furthermore, $\dim A = \dim A_m$ for all maximal ideals m.

__Pf__. First suppose $A \cong \mathbb{k}[x_1, \dots x_d]$ polynomial ring.

$\operatorname{Frac} A = \mathbb{k}(x_1, \dots x_d)$ has transcendence degree d over $\mathbb{k}$

By nullstellensatz, every maximal ideal is of the form

$(x_1 - \alpha_1, \dots, x_d - \alpha_d)$ for some $\alpha_i \in \mathbb{k}$.

So they're all related by linear change of coordinates.

$\Rightarrow \dim A_m$ independent of m.

$\operatorname{gr}_{(x_1, \dots x_d)} A_{(x_1, \dots, x_d)} \cong A \Rightarrow \dim A = d$. ✓

General case: Noether normalization gives:

$B \subset A$ s.t. A is integral over B $\underline{\text{and}}$ $B \cong \mathbb{k}[x_1, \dots, x_d]$ polynomial ring. $\Rightarrow \operatorname{Frac}(A)$ algebraic over $\operatorname{Frac}(B)$.

So both have same transcendence degree over $\mathbb{k}$. $(= d)$

Let $m \subset A$ be maximal ideal.

Then $n = m \cap B$ is also maximal.

Let $P_0 \subsetneq \dots \subsetneq P_r$ be chain of prime ideals in A_m.

Intersect w/ B_n: $p_0 \cap B_n \subsetneq \cdots \subsetneq p_r \cap B_n$.

strict by "incomparability"

$\Rightarrow d = \dim B_n \geq \dim A_m$.

On the other hand any strict chain of prime ideals in B_n comes from intersecting strict chain of prime ideals in A_m by "going-down thm" $\Rightarrow \dim A_m \geq \dim B_n$.

$\Rightarrow \dim A_m = d \quad \forall$ maximal ideals m.

$\Rightarrow \dim A = d$

$\square$