Filtrations

Let A = ring, $I \subset A$ ideal, M = A-module.

A <u>filtration</u> of M is a decreasing chain $\mathcal{F}$ of submodules

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

It is an <u>I-filtration</u> if $IM_i \subseteq M_{i+1}$ for all $i \geq 0$.

<u>stable I-filtration</u> if $IM_i = M_{i+1}$ for $i \gg 0$.

$\Leftrightarrow$ $\exists n$ st. $I^i M_n = M_{n+i}$ for all $i \geq 0$

<u>Ex</u>. $M_n = I^n M$ is a stable I-filtration

<u>Lemma</u>. If $M = M_0 \supseteq M_1 \supseteq \cdots$ are stable I-filtrations,

$M = M_0' \supseteq M_1' \supseteq \cdots$

then $\exists n_0$ st. $M_{n+n_0} \subseteq M_n'$ & $M_{n+n_0}' \subseteq M_n$ for all $n \geq 0$.

<u>pf</u>. $\exists n_0$ s.t. $I^i M_{n_0} = M_{n_0+i}$ for all $i \geq 0$.

$I^i M_{n_0}' = M_{n_0+i}'$

$$\Rightarrow \quad M_{n_0+i} = I^i M_{n_0} \subseteq I^i M \subseteq M_i'$$

$$M_{n_0+i}' = I^i M_{n_0}' \subseteq I^i M \subseteq M_i$$

$\square$

<u>Def</u>. The <u>Rees algebra</u> <u>(blowup algebra)</u> is

$$B_I A := \bigoplus_{n \geq 0} I^n \qquad (I^0 = A)$$

<u>Note</u>: $B_I A$ can be interpreted as subring of $A[t]$

by identifying I^n w/ $I^n t^n = \{ f t^n \mid f \in I^n \}$

$B_I A$ is a $\mathbb{Z}_{\geq 0}$-graded A-algebra via $(B_I A)_d = I^d t^d$

Given an I-filtration $\mathcal{F}$ on M, we define

$$B_{\mathcal{F}} M := \bigoplus_{n \geq 0} M_n$$

$B_{\mathcal{F}}M$ is $\mathbb{Z}_{\geq 0}$-graded $B_I A$-module via

$(B_{\mathcal{F}} M)_d = M_d$ (denote by $M_d t^d$ if needed)

Def. The <u>associated graded ring</u> of I is

$$\mathrm{gr}_I A := \bigoplus_{n \geq 0} I^n / I^{n+1} \cong B_I A / I t^0 \cdot B_I A.$$

If $\mathcal{F}$ is an I-filtration on M, the <u>associated graded module</u> is

$$\mathrm{gr}_{\mathcal{F}} M := \bigoplus_{n \geq 0} M_n / M_{n+1}.$$ ← module over $\mathrm{gr}_I A$

<u>Lemma</u>. If A is noeth ring, then $B_I A$, $\mathrm{gr}_I A$ are noeth.

<u>Pf</u>. I is finitely gen, say by $f_1, \ldots f_r \in I$.

$\Rightarrow f_1 t, \ldots, f_r t \in B_I A$ generate it as A-algebra.

Hilbert basis thm $\Rightarrow$ $B_I A$ is noeth.

$\mathrm{gr}_I A$ is a quotient of $B_I A$, so also noeth. $\square$

<u>Prop</u>. If A is noeth ring, M is f.g A-module.

$I \subset A$ ideal, $\mathcal{F} = I$-filtration on M.

Then: $\mathcal{F}$ is I-stable $\Leftrightarrow$ $B_{\mathcal{F}} M$ is a f.g. $B_I A$-module.

<u>Pf</u>. Suppose $\mathcal{F}$ is I-stable. $\exists\, n$ s.t. $M_{n+i} = I^i M_n$ for all $i \geq 0$.

$\Rightarrow B_{\mathcal{F}} M$ is generated as $B_I A$-module by $M_0 t^0, \ldots, M_n t^n$

A noeth $\Rightarrow M_i$ is f.g A-module for all i

Pick finite list of generators for $M_0, \ldots M_n$ ⇝ finite list of generators of $B_{\mathcal{F}} M$.

Now suppose $B_{\mathcal{F}} M$ is a f.g. $B_I A$-module.

Pick finite list of generators (may assume homogeneous) $\rightarrow m_i$

Let $n =$ max degree of any of these generators.

Every element of $M_{n+i}t^{n+i}$ $(i \geq 0)$ is a linear combination of the form $\sum_j f_j m_j$ where $m_j \in M_{d(j)}$ and $f_j \in I^{n+i-d(j)}$. If $d(j) < n$, then can rewrite $f_j m_j$ as a sum of products of elements in I^i & M_n, so this implies

$I^i M_n = M_{n+i} \Rightarrow \mathcal{F}$ is I-stable. $\square$

Cor. If $\mathcal{F}$ is I-stable, then $gr_{\mathcal{F}}(M)$ is a f.g. $gr_I(A)$-module

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Cor. (Artin-Rees Lemma) $A =$ noeth ring, $M =$ f.g. A-module. $I \subset A$ ideal, $\mathcal{F}$ is stable I-filtration on M.

$M' \subseteq M$ is A-submodule.

Then: the filtration $\mathcal{F}'$ on M' defined by $M'_n = M' \cap M_n$ is I-stable.

Pf. $B_{\mathcal{F}}M$ is a f.g. $B_I A$-module.

$B_I A$ noeth $\Rightarrow B_{\mathcal{F}}M$ noeth.

$B_{\mathcal{F}'}M'$ is $B_I A$-submodule of $B_{\mathcal{F}}M \Rightarrow B_{\mathcal{F}'}M'$ is f.g. $B_I A$-module.

$\Rightarrow \mathcal{F}'$ is I-stable. $\square$

Cor. A = noeth ring, M = f.g. A-module. $M' \subseteq M$ A-submodule. $I \subset A$ ideal.

Thm: $\exists k$ s.t. if $n \geq k$ then

$$(I^n M) \cap M' = I^{n-k}((I^k M) \cap M')$$

Pf. $M_n = I^n M$ is a stable I-filtration of M.

By Artin-Rees, $M_n' = (I^n M) \cap M'$ is also stable I-filtration.

$\square$