Graded rings & Hilbert series

$(\Gamma, +)$ = abelian semigroup (+ is associative, commutative binary operation w/ unit 0)

Def. A Γ-grading on ring A is a direct sum decomposition
$$A = \bigoplus_{\gamma \in \Gamma} A_\gamma \quad \text{s.t. if} \quad f \in A_\gamma, f' \in A_{\gamma'}, \text{ then } ff' \in A_{\gamma+\gamma'}.$$

If $f \in A_\gamma$ for some γ, f is homogeneous.

Given A-module M, a Γ-grading on M is direct sum decomposition
$$M = \bigoplus_{\gamma \in \Gamma} M_\gamma \quad \text{s.t. if} \quad f \in A_\gamma,\ x \in M_{\gamma'}, \text{ then } fx \in M_{\gamma+\gamma'}.$$

Ex. $\Gamma = (\mathbb{Z}_{\geq 0}, +)$, or $\Gamma = (\mathbb{Z}, +)$ or $\Gamma = (\mathbb{Z}^d_{\geq 0}, +)$ ...

typical example

Notation: For $\Gamma = \mathbb{Z}_{\geq 0}$, $M_{\geq d} = \bigoplus_{n \geq d} M_n$ (truncation)

For $\Gamma = \mathbb{Z}$, $M(d) = M$ w/ shifted grading
$$M(d)_n = M_{d+n}$$

A homomorphism between graded modules $f: M \to N$ is of degree γ if $f(M_{\gamma'}) \subseteq N_{\gamma+\gamma'}$ for all $\gamma' \in \Gamma$.

If $\Gamma = \mathbb{Z}$, we can interpret degree d map as a degree 0 map $f: M(-d) \to N$.

For M, N graded, define (for $\Gamma = \mathbb{Z}_{\geq 0}$)
$$(M \oplus N)_n = M_n \oplus N_n, \quad (M \otimes N)_n = \bigoplus_{i=0}^{n} M_i \otimes N_{n-i}$$

Ex. A = ring, poly ring $A[x_1, \dots, x_n]$ is $\mathbb{Z}_{\geq 0}$-graded by
$$A[x_1, \dots x_n]_d = A\text{-submodule spanned by } x_1^{p_1} \cdots x_n^{p_n} \text{ w/ } p_1 + \dots + p_n = d$$

From now on, fix noeth ring A w/ $\mathbb{Z}_{\geq 0}$-grading $A = \bigoplus_{d \geq 0} A_d$.

A_0 is quotient of A by $A_+ = \bigoplus_{d \geq 1} A_d$ (so A_0 is noeth)

Def. An <u>additive function</u> λ on class of finitely gen. A_0-modules is a function (taking values in integers) s.t. given exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

we have $\lambda(M_2) = \lambda(M_1) + \lambda(M_3)$.

Note: given any exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to \cdots \to M_d \to 0,$$

we have $\lambda(M_1) - \lambda(M_2) + \lambda(M_3) + \cdots + (-1)^d \lambda(M_d) = 0$.

Ex. If A_0 is artinian, then $\lambda = \ell$ (length) is additive.
(If A_0 = field, $\lambda = \dim$ is additive).

Let M be $\mathbb{Z}_{\geq 0}$-graded and finitely generated as A-module.
Then for all n, M_n is a finitely gen. A_0-module: $M_{\geq n}$ is a A-submodule of M (hence f.g. since A noeth). Take finite set of homogeneous generators. Those of degree n give A_0-module generators for M_n. $\Rightarrow$ $\lambda(M_n)$ is defined.

Def. <u>Hilbert series</u> of M (with respect to λ) is formal power series (in variable t)

$$H_M(\lambda, t) := \sum_{n \geq 0} \lambda(M_n) t^n \in \mathbb{Z}[[t]]$$

(If A_0 = field, $\lambda = \dim$, then write $H_M(t)$ instead of $H_M(\lambda, t)$.)

Fact: A formal power series is invertible if and only if its constant term is invertible. Hence $\frac{1}{1-t^d}$ makes sense as formal power series.

Thm. Suppose A is generated as A_0-algebra by homog. elements of degrees $d_1, \dots, d_r$. Then for every $\mathbb{Z}_{\geq 0}$-graded f.g. A-module M, $\exists\ h_M(t) \in \mathbb{Z}[t]$ s.t.

$$H_M(\lambda, t) = \frac{h_M(t)}{(1-t^{d_1})\cdots(1-t^{d_r})}$$

pf. Induction on $r = \#$ generators of A.

If $r=0$, then $A = A_0$, and $M_n = 0$ for $n \gg 0$.

Then $H_M(\lambda, t) \in \mathbb{Z}[t]$, so take $h_M(t) = H_M(\lambda, t)$.

Else, suppose $r > 0$, and statement holds for $r-1$ generators.

let $x_1, \dots x_r$ be homog gens for A of degrees $d_1, \dots d_r$.

Multiplication by x_r is a degree d_r map $M \to M$,

let $K = \{m \mid x_r m = 0\}$. Then have exact sequence

$$0 \to K(-d_r) \to M(-d_r) \xrightarrow{\cdot x_r} M \to M/x_rM \to 0$$

A noeth $\Rightarrow$ K finitely gen; since λ additive, have

$$H_{K(-d_r)}(\lambda, t) - H_{M(-d_r)}(\lambda, t) + H_M(\lambda, t) - H_{M/x_rM}(\lambda, t) = 0.$$

$$\Rightarrow\ t^{d_r} H_K(\lambda, t) - t^{d_r} H_M(\lambda, t) + H_M(\lambda, t) - H_{M/x_rM}(\lambda, t) = 0$$

$$\Rightarrow\ H_M(\lambda, t) = \frac{H_{M/x_rM}(\lambda, t) - t^{d_r} H_K(\lambda, t)}{1-t^{d_r}}.$$

Note: x_r annihilates K & M/x_rM, so both are f.g. over $A/(x_r)$ ← f.g. A_0-algebra via cosets of $x_1, \dots x_{r-1}$

$\Rightarrow$ $\exists\ h_{M/x_rM}(t),\ h_K(t) \in \mathbb{Z}[t]$ s.t.

$$H_{M/x_rM}(\lambda, t) = \frac{h_{M/x_rM}(t)}{(1-t^{d_1})\cdots(1-t^{d_{r-1}})} \; ; \; H_K(\lambda, t) = \frac{h_K(t)}{(1-t^{d_1})\cdots(1-t^{d_{r-1}})}$$

Finally, take $h_M(t) = h_{M/x_rM}(t) - t^{d_r} h_K(t)$. $\square$

<u>Ex.</u> $A_0 = \text{field}$, $\lambda = \dim$, $A = A_0[x_1, \ldots x_r]$

Then $A \cong A_0[x_1] \otimes \cdots \otimes A_0[x_r]$

$$\Rightarrow \quad H_A(t) = H_{A_0[x_1]}(t) \cdots H_{A_0[x_r]}(t).$$

$$H_{A_0[x_i]}(t) = \sum_{n \geq 0} t^{nd_i} = \frac{1}{1-t^{d_i}}$$

$$\Rightarrow \quad H_A(t) = \frac{1}{(1-t^{d_1}) \cdots (1-t^{d_r})}.$$

More generally, if A_0 artinian and $\lambda = \text{length}$,

Then $$H_A(\lambda, t) = \frac{\ell(A_0)}{(1-t^{d_1})\cdots(1-t^{d_r})}.$$

<u>Hilbert function</u> of M is $n \to \lambda(M_n)$.

First suppose $d_i = 1$ for all i.

<u>Prop.</u> Let (a_n) be sequence in a field. Then $\exists$ polynomial $f(t)$ s.t.

$$\sum_{n \geq 0} a_n t^n = \frac{f(t)}{(1-t)^r} \iff \exists \text{ polynomial } \alpha(x) \text{ of degree}$$

$\leq r-1$ s.t. $\alpha(n) = a_n$ for $n \gg 0$.

In this case, $1 + \deg \alpha(x) = \text{order of pole at } t=1 \text{ of } \frac{f(t)}{(1-t)^r}$

i.e., $r -$ mult of root $t=1$ of $f(t)$.

Pf. First suppose $\sum_{n\geq 0} a_n t^n = \frac{f(t)}{(1-t)^r}$ for some $f(t) = \sum_{i=0}^{N} f_i t^i$.

Note: $(1-t)^{-r} = \left(\sum_{n\geq 0} t^n\right)^r = \sum_{n\geq 0} \binom{n+r-1}{r-1} t^n$:

the coeff of t^n in $\left(\sum_{d\geq 0} t^d\right)^r$ is # of tuples $(a_1,\dots,a_r)$

$a_i \in \mathbb{Z}_{\geq 0}$ s.t. $a_1+\dots+a_r = n$

Each r-tuple is encoded by sequence of n a's and $r-1$ $+$'s:

a_1 a's followed by $+$, followed by a_2 a's, ...

Ex. $(4,2,3) \leftrightarrow aaaa+aa+aaa$

For all $n \geq \deg f$, we have

$$a_n = \sum_{i=0}^{N} f_i \binom{r+n-i-1}{r-1}$$

If $g(x) := \frac{(r+x-1)(r+x-2)\cdots(x+1)}{(r-1)!}$, then

$g(n) = \binom{r+n-1}{r-1}$ for $n \geq 0$. and $\deg g(x) = r-1$.

We take $\alpha(x) = \sum_{i=0}^{N} f_i\, g(x-i)$.

Now suppose $\exists\ \alpha(x)$ of degree $\leq r-1$ s.t. $\alpha(n) = a_n$ for $n \gg 0$.

Prove result by induction on r.

If $r=1$: then a_n constant for $n \gg 0$, so

$$\sum_{n\geq 0} a_n t^n = \text{constant}\cdot \sum_{n\geq 0} t^n + \text{polynomial in } t.$$

$$= \frac{\text{constant}}{1-t} + \text{poly}(t).$$

If $r>1$, let $b_n = a_n - a_{n-1}$ (set $a_{-1}=0$)

Then $\alpha(x)-\alpha(x-1)$ is a poly. of degree $\leq r-2$ which agrees w/ b_n for $n\gg 0$. So by induction,

$$(1-t)\sum_{n\geq 0} a_n t^n = \sum_{n\geq 0} b_n t^n = \frac{f(t)}{(1-t)^{r-1}} \text{ for some } f(t).$$

Now divide by $1-t$.

Last statement: let $d=$ order of pole at $t=1$ of $\sum_{n\geq 0} a_n t^n$.

So $\exists f(t)$ s.t. $f(1)\neq 0$ and $\sum_{n\geq 0} a_n t^n = \frac{f(t)}{(1-t)^d}$.

$\Rightarrow \exists\ \alpha(x)$ of degree $\leq d-1$ s.t. $\alpha(n)=a_n$ for $n\gg 0$.

If $\deg \alpha(x)\leq d-2$ then $\exists g(t)$ s.t. $\sum_{n\geq 0} a_n t^n = \frac{g(t)}{(1-t)^{d-1}}$

$$\Rightarrow \frac{f(t)}{(1-t)^d} = \frac{g(t)}{(1-t)^{d-1}} \Rightarrow 1-t \text{ divides } f(t) \to\!\leftarrow.$$

$\Rightarrow \deg \alpha(x) = d-1.$ $\square$

Cor. If A is generated by degree 1 elements as A_0-algebra, then for every $\mathbb{Z}_{\geq 0}$-graded f.g. A-module M, $\exists$ polynomial $p_M(t)$ of degree $\leq r-1$ s.t.

$$\lambda(M_n) = p_M(n) \text{ for } n\gg 0.$$

Hilbert polynomial of M.

General $d_1,\dots,d_r$: A function g defined on $\mathbb{Z}_{\geq 0}$ is a quasi-polynomial of period m if $\exists$ polynomials $p_0,\dots,p_{m-1}$ s.t. $g(n)=p_i(n)$ whenever $i\equiv n \pmod{m}$.

Degree of $g = \max(\deg p_i)$.

Cor. If A is gen. by homog. elements of degrees $d_1, \dots d_r$ as A_0-algebra then for every $\mathbb{Z}_{\geq 0}$-graded f.g. A-module M, $\exists$ quasi-polynomial $p_M(x)$ of period $\mathrm{lcm}(d_1, \dots d_r)$ and degree $\leq r-1$ s.t.

$$\lambda(M_n) = p_M(n) \quad \text{for } n \gg 0.$$