Math 200C, Spring 2022
Homework 1
Due: April 8 11:59PM via Gradescope

Please do not search for solutions. I would rather help you directly (via office hours or Discord) so that I can calibrate explanations in the notes and lecture. You are free to work with other students, but solutions must be written in your own words. Please cite any sources (beyond the course materials) that you use or any people you collaborated with.

This covers the material in Section 2 of the notes (lectures 2–4).

(1) Let $f\colon A \to B$ be a ring homomorphism and let $S \subset A$ be a multiplicative subset. Define $T = f(S)$. Let M be a B-module. Construct an $S^{-1}A$-linear isomorphism between $S^{-1}M$ (where M is considered an A-module via f) and $T^{-1}M$ (considered an $S^{-1}A$-module via the map $S^{-1}A \to T^{-1}B$ given by $a/s \mapsto f(a)/f(s)$).

(2) Let M be an A-module. Define the **support** of M to be

$$\mathrm{Supp}(M) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid M_{\mathfrak{p}} \neq 0\}.$$

   (a) Show that $\mathrm{Supp}(M) \subseteq V(\mathrm{Ann}(M))$ (recall that $\mathrm{Ann}(M) = \{x \in A \mid xm = 0 \text{ for all } m \in M\}$ and here we use the notation from §1.4), and that equality holds if M is finitely generated.

   (b) Let N be another A-module. Show that $\mathrm{Supp}(M \otimes_A N) \subseteq \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$, and that equality holds if M and N are finitely generated (Exercise 2.3 of Atiyah–Macdonald may be helpful here.)

(3) Let M be an A-module and let $S \subset A$ be a multiplicative set. Show that the natural map $M \to S^{-1}M$ given by $m \mapsto m/1$ is a bijection if and only if, for all $x \in S$, the multiplication map $m \mapsto xm$ is an isomorphism of M.

(4) Let A be a ring and let $0 \to X \to Y \to Z \to 0$ be a chain complex of A-modules. Show that the following are equivalent:

   (a) $0 \to X \to Y \to Z \to 0$ is exact.

   (b) $0 \to X_{\mathfrak{p}} \to Y_{\mathfrak{p}} \to Z_{\mathfrak{p}} \to 0$ is exact for all prime ideals $\mathfrak{p} \subset A$.

   (c) $0 \to X_{\mathfrak{m}} \to Y_{\mathfrak{m}} \to Z_{\mathfrak{m}} \to 0$ is exact for all maximal ideals $\mathfrak{m} \subset A$.

(5) Let M be an A-module. Suppose that for each maximal ideal $\mathfrak{m}$ of A, there exists $f \notin \mathfrak{m}$ such that M_f is a finitely generated A_f-module. Pick one such element with this property and call it $f_{\mathfrak{m}}$.

   (a) Show that there is a finite subset $\{f_1, \dots, f_r\}$ of $\{f_{\mathfrak{m}}\}$ that generates the unit ideal.

   (b) Use the generators for $M_{f_1}, \dots, M_{f_r}$ to get a finite generating set for M.

1. Extra problems (don't submit)

(6) Let A be a ring with multiplicative set $S \subset A$. Let M, N be A-modules and assume that M is finitely presented. Construct an $S^{-1}A$-linear isomorphism between $\mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$ and $S^{-1}\mathrm{Hom}_A(M, N)$. (See Proposition 2.10 of Eisenbud for a more general statement and proof.)

(7) Atiyah–Macdonald, Exercise 3.12

(8) Atiyah–Macdonald, Exercise 3.13