Math 190A, Fall 2022
Homework 5
Due: **Monday**, November 21, 2022 11:59PM via Gradescope
(late submissions allowed up until November 22, 2022 11:59PM with -25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Let X be a topological space and let $\sim$ be an equivalence relation on X with quotient map $\pi\colon X \to X/{\sim}$.
  (a) If $A \subseteq X/{\sim}$ is a subspace and C is a connected component of A, prove that $\pi^{-1}(C)$ is a union of connected components of $\pi^{-1}(A)$.
  (b) Prove that if X is locally connected, then so is $X/{\sim}$. Hint at end.

(2) Finish the details of Examples 4.1.17 and 4.1.20 to prove that $\mathbf{U}(n)$ and $\mathbf{CP}^n$ are compact.

(3) **Tube lemma.** Let A be a topological space and let B be a compact space. Suppose there exists $a \in A$ and an open set $U \subseteq A \times B$ such that $\{a\} \times B \subseteq U$. Then there exists a neighborhood V_a of a in A such that $V_a \times B \subseteq U$.

Convince yourself that we actually proved this statement in the proof of Proposition 4.1.12. You don't need to write anything down.
  (a) If A is any space and B is compact, prove that for any closed subset $Z \subseteq A \times B$, $\pi_1(Z) \subseteq A$ is closed. Hint at end.
  (b) The result can fail for $\pi_2\colon A \times B \to B$. Concretely, find an example (with an explanation) of a closed subset Z of $\mathbf{R} \times [0, 1]$ such that $\pi_2(Z)$ is not closed.

(4) Let X be the set of $k \times n$ real matrices.
  (a) Define the subset Z of $X \times (\mathbf{R}^n \setminus \{\mathbf{0}\})$ by
$$Z = \{(A, x) \mid Ax = 0\}$$
where Ax is the usual product of the matrix A with the column vector x. Prove that Z is a closed subset.
  (b) The projection $\pi_1(Z)$ is the set of matrices with a nonzero kernel element. Prove that $\pi_1(Z)$ is a closed subset of X. Hint at end.

(5) (a) Let X be a limit point compact space. If $A \subseteq X$ is a closed subspace, prove that A is also limit point compact.
  (b) Let $X = \mathbf{Z}$ be the set of integers with the discrete topology and let $Y = \{0, 1\}$ with the indiscrete topology. Prove that X is not limit point compact but that $X \times Y$ is limit point compact (actually you can prove something stronger: every nonempty subset of $X \times Y$ has a limit point).

[Remark: Since X is the image of $X \times Y$ under the projection map, this example shows that the image of a limit point compact space under a continuous function does not have to be limit point compact. It also shows that $X \times Y$ is not compact, so that limit point compact does not imply compact.]

Hints

1b: Use Proposition 3.3.14.
3a: If $\pi_1(Z) = A$, nothing to show. Otherwise, try to apply the tube lemma to $(A \times B) \setminus Z$.
4b: Find a compact subspace $Y \subseteq \mathbf{R}^n \setminus \{0\}$ such that $\pi_1(Z) = \pi_1(Z \cap (X \times Y))$.

Optional problems (don't turn in)

(6) Finish Example 4.1.20: prove that $\mathbf{Gr}(k, \mathbf{R}^n)$ and $\mathbf{Gr}(k, \mathbf{C}^n)$ are compact.

(7) This exercise outlines how to prove that a general product of compact spaces is compact.

Let P be a partially ordered set (poset). Some definitions:

- A subset Q of P is called a **chain** if given $x, y \in Q$, we have either $x \leq y$ or $y \leq x$.
- Given a subset Q of P, an **upper bound** of Q is any element $u \in P$ such that $u \geq x$ for all $x \in Q$.
- An element $z \in P$ is **maximal** if, for all $x \in P$, $x \geq z$ implies that $x = z$.

Zorn's lemma states that if P is nonempty and every chain of P has an upper bound, then there exist maximal elements.

(a) Let X be a topological space which is not compact. Define a poset P whose elements are the open coverings of X which do not have finite subcoverings. Order them by refinement: $\{U_i\}_{i \in I} \leq \{V_j\}_{j \in J}$ if for every $i \in I$, there exists $j \in J$ such that $U_i = V_j$. Use Zorn's lemma to conclude that P has a maximal element.

(b) Let X be a topological space with a subbasis S such that every open covering of X by elements of S has a finite subcovering. Prove that X is compact as follows:

(i) Assume X is not compact and use (a) to get a maximal covering $\{U_i\}_{i \in I}$ with no finite subcovering.

(ii) Define $J = \{i \in I \mid U_i \in S\}$. Show that $\bigcup_{j \in J} U_j \neq X$.

(iii) Pick $x \in X \setminus \bigcup_{j \in J} U_j$. There exists U_i such that $x \in U_i$ and there exist $V_1, \ldots, V_n \in S$ such that $x \in V_1 \cap \cdots \cap V_n \subseteq U_i$. Show that $\{V_k\} \cup \{U_i\}_{i \in I}$ has a finite subcovering for each $k = 1, \ldots, n$ and use this to produce a finite subcovering of $\{U_i\}_{i \in I}$.

[This is the *Alexander subbasis theorem.*]

(c) Let $\{X_i\}_{i \in I}$ be a collection of compact spaces. Recall that the product $X = \prod_{i \in I} X_i$ has a subbasis $S = \{\pi_j^{-1}(U_j)\}$ as j ranges over all elements of I and U_j ranges over all open subsets of X_j. Prove that every open covering of X by elements in S has a finite subcovering and conclude that X is compact.

[This is *Tychonoff's theorem.*]