Math 190A, Fall 2022
Homework 4
Due: November 4, 2022 11:59PM via Gradescope
(late submissions allowed up until November 5, 2022 11:59PM with −25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Let X be a topological space and let $\mathbf{Z}_{\geq 0}$ be the set of nonnegative integers. For each $n \in \mathbf{Z}_{\geq 0}$, let A_n be a connected subspace of X. If $A_n \cap A_{n+1} \neq \varnothing$ for all n, prove that $\bigcup_{n \in \mathbf{Z}_{\geq 0}} A_n$ is connected.

(2) Let X and Y be connected spaces and let $A \subsetneqq X$ and $B \subsetneqq Y$ be proper subsets. Prove that $(X \times Y) \setminus (A \times B)$ is connected. [See next page for some hints.]

(3) (a) Prove that no two of these subspaces of $\mathbf{R}$ are homeomorphic: $[0,1]$, $[0,1)$, $(0,1)$.
(b) If $n \geq 2$, prove that $\mathbf{R}$ is not homeomorphic to $\mathbf{R}^n$.

(4) For each of the following spaces, find all of its connected components (with a proof that your statement is correct, of course). As usual, $\mathbf{R}$ is the set of real numbers, and $\mathbf{Q}$ is the set of rational numbers.
(a) $\mathbf{R} \setminus \mathbf{Q}$
(b) $X = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \mid x_i \neq x_j \text{ for } i \neq j\}$. In words, X is the space of n-tuples where all entries are distinct.

(5) Let X be a topological space and let $\{U_i\}_{i \in I}$ be an open covering (reminder: this means that each U_i is open in X and $\bigcup_{i \in I} U_i = X$). Prove that X is locally connected if and only if U_i is locally connected for all $i \in I$.

Hints

(3) Hint 1: You'll want to modify the ideas in the proof of Proposition 3.1.19 that a product of connected space is connected.

Hint 2: Pick $a \in X \setminus A$ and $b \in Y \setminus B$. Prove that (using the notation from 3.1.19)

$$(X \times Y) \setminus (A \times B) = \left(\bigcup_{x \in X \setminus A} T_{x,b} \right) \cup \left(\bigcup_{y \in Y \setminus B} T_{a,y} \right).$$

Optional problems (don't turn in)

(6) Let X be a metrizable space with countably many elements. Prove that every connected component of X consists of one element.

(7) Let I be an index set and let X_i be a connected space for each $i \in I$. Define $X = \prod_{i \in I} X_i$. Choose one point $a_i \in X_i$ for each $i \in I$.
  - (a) For each finite subset $J \subseteq I$, let X_J be the subspace of X consisting of tuples $(x_i)_{i \in I}$ such that $x_j = a_j$ for all $j \in J$. Prove that X_J is connected.
  - (b) Let S be the collection of finite subsets of I and define $Y = \bigcup_{J \in S} X_J$. Prove that Y is connected.
  - (c) Finally, prove that $\overline{Y} = X$, and conclude that X is connected.