Math 190A, Fall 2022
Homework 2
Due: October 14, 2022 11:59PM via Gradescope
(late submissions allowed up until October 15, 2022 11:59PM with -25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Let $f\colon X \to Y$ be a function between topological spaces. Let S be a subbasis for Y. Prove that f is continuous if and only if for all $s \in S$, $f^{-1}(s)$ is open in X.

(2) As usual, define

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\},$$
$$(c, \infty) = \{x \in \mathbf{R} \mid c < x\},$$
$$(-\infty, d) = \{x \in \mathbf{R} \mid x < d\},$$

and give them all the subspace topology from $\mathbf{R}$. Prove that

$$(a, b), (c, \infty), (-\infty, d), \mathbf{R}$$

are all homeomorphic to each other for any a, b, c, d such that $a < b$. You are free to use any standard functions from calculus and you do not have to reprove that they are continuous.

(3) Let X, Y be topological spaces and let $A \subseteq X$ and $B \subseteq Y$ be subsets.
   (a) If A is closed in X and B is closed in Y, prove that $A \times B$ is closed in $X \times Y$.
   (b) In general, prove that $\overline{A \times B} = \overline{A} \times \overline{B}$ as subsets of $X \times Y$. To be more precise about the notation, we want $\mathrm{Cl}_{X\times Y}(A \times B) = \mathrm{Cl}_X(A) \times \mathrm{Cl}_Y(B)$.

(4) Let $f\colon X \to Y$ be a continuous function. Define $g\colon X \to X \times Y$ by $g(x) = (x, f(x))$. Prove that g is an embedding.

(5) Let A, B, C, D be topological spaces and let $f\colon A \to C$ and $g\colon B \to D$ be continuous functions. Define $h\colon A \times B \to C \times D$ by $h(a, b) = (f(a), g(b))$. Prove that h is continuous.

Optional problems (don't turn in)

(6) Let $f\colon X \to Y$ be a function between topological spaces. Prove that the following are equivalent:
   (a) f is continuous.
   (b) For every subset $B \subseteq Y$, we have $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.
   (c) For every subset $B \subseteq Y$, we have $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$.

(7) Let $X = \mathrm{S}^1 \setminus \{(0,1)\}$, given the subspace topology from $\mathbf{R}^2$. Define a function $f\colon X \to \mathbf{R}$ as follows. Given $(a,b) \in X$, there is a unique line through $(0,1)$ and (a,b) which intersects the x-axis at a point $(0,c)$; define $f(a,b) = c$. Prove that f is a homeomorphism.

   [Hint: You can find explicit formulas for f and its inverse and then conclude that both are continuous using standard calculus facts.]

   Find and prove the analogous statement about $\mathrm{S}^n \setminus \{(0,0,\ldots,0,1)\}$ and $\mathbf{R}^n$.

(8) Let I be an index set and suppose for each $i \in I$, we have a topological space X_i and a basis B_i for the topology on X_i. Define B to be the collection of $\prod_{i \in I} b_i$ where $b_i \in B_i \cup \{X_i\}$ and $b_i = X_i$ for all but finitely many $i \in I$. Prove that B is a basis for the product topology on $\prod_{i \in I} X_i$.