

Math 188, Fall 2022

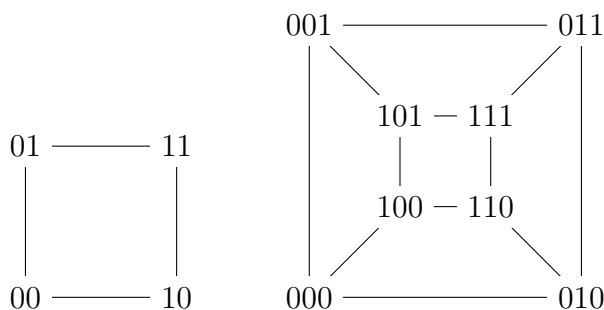
Homework 5

Due: November 19, 2022 11:59PM via Gradescope

(late submissions allowed up until November 20, 2022 11:59PM with -25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

- (1) Let B_n be the set of binary strings of length n , i.e., of words of length n in the alphabet $\{0, 1\}$. Define a simple graph H_n whose vertex set is B_n and where two binary strings are connected by an edge if and only if they differ in exactly one position. Here are H_2 and H_3 :



Let V be the real vector space with basis $\{v_x \mid x \in B_n\}$. It's not convenient to pick an ordering of the vertices to write down the adjacency matrix. Instead, we will think of the adjacency matrix A of H_n as a linear operator on V :

$$A \sum_{x \in B_n} c_x v_x = \sum_{x \in B_n} c_x \sum_{\substack{y \text{ such that} \\ \{y, x\} \text{ is an edge}}} v_y.$$

- (a) For each $x \in B_n$, define $E_x \in V$ by

$$E_x = \sum_{y \in B_n} (-1)^{x \cdot y} v_y$$

where $x \cdot y = x_1 y_1 + \dots + x_n y_n$. Show that this is an eigenvector for A with eigenvalue $n - 2|x|$ where $|x|$ is the number of x_i equal to 1.

- (b) For $x \in B_n$ and a non-negative integer d , give a formula for the number of closed walks of length d in H_n beginning at x . Hint at end.
[You may assume without proof that $\{E_x \mid x \in B_n\}$ is linearly independent.]

- (2) (a) Fix a positive integer k . Construct a directed graph for which walks (between certain vertices) can be interpreted as binary strings with exactly k zeroes. Explain clearly how this interpretation works, including how the length of the walk relates to the length of the binary string.
- (b) Construct a directed graph for which walks (between certain vertices) can be interpreted as binary strings in which no symbol ever appears 3 times in a row. Explain clearly how this interpretation works, including how the length of the walk relates to the length of the binary string.

Hint at end.

- (3) (a) We have n distinguishable tables. We want to paint each one either red, blue, or green such that an odd number of them are red and an even number of them are blue. How many ways can this be done?
- (b) Continuing with (a), we add the colors white and yellow, but the total number of tables which are white or yellow must be even (in symbols: $\#$ white tables + $\#$ yellow tables is even). How many ways are there to choose colors?
- (4) Let $b_{n,k}$ be the number of set partitions of $[n]$ with k blocks such that every block has an even (and positive) number of elements and let b_n be the same, but with no restriction on the number of blocks.
- (a) Find a formula for the EGF $B_k(x) = \sum_{n \geq 0} b_{n,k} \frac{x^n}{n!}$.
- (b) Find a formula for the EGF $B(x) = \sum_{n \geq 0} b_n \frac{x^n}{n!}$.
- (5) Let a_n be the number of functions $f: [n] \rightarrow [n]$ such that $f \circ f = f$. Find a simple formula for the EGF $A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$. Hint at end. [Note that it's possible to find a formula for a_n using basic methods, but I suggest that you don't use that formula and just use the general methods of EGF discussed in class.]

HINTS

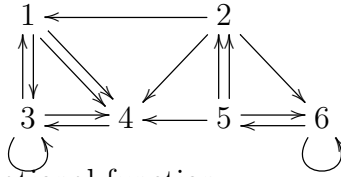
1b: We saw in lecture how to compute the total number of closed walks, but not for an individual vertex. But note that H_n has a lot of symmetry.

2: For these kinds of problems, you want to think of a vertex as a “memory state” so that being at the i th vertex in a walk tells you some information about the first i letters of the binary string without having to know which vertices were previously visited.

5: Encode a function $f: [n] \rightarrow [n]$ as a directed graph with vertices $[n]$ and an edge $i \rightarrow j$ if $f(i) = j$. If $f \circ f = f$, what do its connected components look like?

OPTIONAL PROBLEMS (DON'T TURN IN)

- (6) Explain why it is impossible to have a directed graph for which walks of length $2n$ can be interpreted as balanced sets of n pairs of parentheses.
- (7) Let A be the adjacency matrix of the following directed graph:



Express $F_{A;1,4}(x)$ as a rational function.

- (8) Compute G_3 in Example 4.11.
- (9) Given a sequence $(a_n)_{n \geq 0}$ we define its q -EGF to be

$$A(x) = \sum_{n \geq 0} a_n \frac{x^n}{[n]_q!}.$$

What is the q -analogue of Proposition 5.2?

- (10) Let n be a positive integer. Given a group of n people, we want to divide them into nonempty committees and choose a leader for each committee, and also choose one of the committees to be in charge of all of the others. Let h_n be the number of ways to do this and set $h_0 = 1$. Give a simple expression for the exponential generating function $H(x) = \sum_{n \geq 0} \frac{h_n}{n!} x^n$.
- (11) Let h_n be the number of bijections $f: [n] \rightarrow [n]$ with the property that $f \circ f \circ f$ is the identity function. Give a simple expression for the exponential generating function $H(x) = \sum_{n \geq 0} \frac{h_n}{n!} x^n$.
- (12) Given $G(x)$ with $G(0) \neq 0$, define its **logarithmic derivative** to be $\mathcal{L}(G) = \frac{DG(x)}{G(x)}$.
- (a) Show that for any $F(x)$, we have $\mathcal{L}(\exp(F(x))) = DF(x)$.
- (b) Show that if $G_1(0) = G_2(0) = 0$, then $\mathcal{L}(G_1(x)G_2(x)) = \mathcal{L}(G_1(x)) + \mathcal{L}(G_2(x))$.
- (c) Let a_n be the number of involutions of size n and let $A(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}$. From Example 5.11, we have $A(x) = \exp(x + \frac{x^2}{2})$. Apply \mathcal{L} to prove for all $n \geq 0$ that

$$a_{n+2} = a_{n+1} + (n+1)a_n.$$