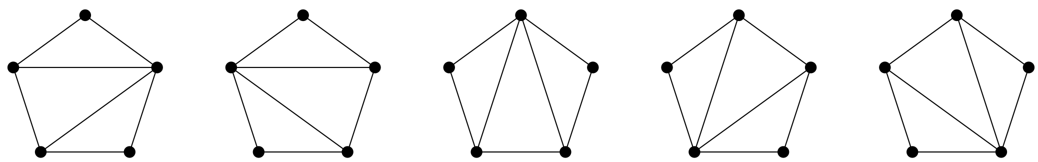
Math 188, Fall 2022
Homework 3
Due: October 29, 2022 11:59PM via Gradescope
(late submissions allowed up until October 30, 2022 11:59PM with -25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Let n be a positive integer. Show that the number of ways of triangulating (i.e., drawing diagonals between vertices that do not intersect except at vertices so that the regions are all triangles) a convex polygon with $(n+2)$ vertices is the nth Catalan number C_n. By convention, the "2-gon" and triangle both have exactly one triangulation and here are the 5 triangulations of a pentagon:



(2) Consider the following variation of counting balanced parentheses. We have a new symbol $*$. Let a_n be the number of length n strings consisting of left/right parentheses and $*$ such that the result of deleting all of the $*$'s is a balanced set of parentheses ($a_0 = 1$). Let $A(x) = \sum_{n \geq 0} a_n x^n$. Find polynomials $a(x), b(x), c(x)$ in x, not all identically 0, such that

$$a(x)A(x)^2 + b(x)A(x) + c(x) = 0.$$

(3) Let n be a positive integer. Consider the equation

$$x_1 + x_2 + \cdots + x_8 = 2n.$$

For each of the following conditions, how many solutions are there? Give as simple of a formula as possible. (Each part is an independent problem.)

(a) The x_i are non-negative even integers.
(b) The x_i are positive odd integers.
(c) The x_i are non-negative integers and $x_8 \leq 9$.

(4) Let k, n be positive integers such that $k \geq n$.

(a) Show that

$$\sum_{(a_1,\dots,a_n)} a_1 a_2 \cdots a_n = \binom{n+k-1}{k-n}$$

where the sum is over all compositions of k into n parts. (Hint at end.)

(b) Show that

$$\sum_{(a_1,\dots,a_n)} 2^{a_2-1} 3^{a_3-1} \cdots n^{a_n-1} = S(k,n)$$

where the sum is over all compositions of k into n parts.

(5) (a) Give a closed formula for the number of pairs of subsets S, T of $[n]$ such that $S \subsetneq T$ (i.e., $S \subseteq T$ and $S \neq T$).

(b) Give a closed formula for the number of k-tuples of subsets $(S_1, \dots, S_k)$ of $[n]$ such that $\bigcup_{i=1}^k S_i = [n]$.

Hints

4a: Consider the product $(\sum_{a_1 \geq 1} a_1 x^{a_1}) \cdots (\sum_{a_n \geq 1} a_n x^{a_n})$. Same idea works for 4b.

Optional problems (don't turn in)

(6) Give a closed formula for the number of k-tuples of subsets $(S_1, \ldots, S_k)$ of $[n]$ such that $S_i \subseteq S_{i+1}$ for $i = 1, \ldots, k-1$.

(7) What is the total number of parts of all compositions of k?

[For example, when $k = 2$, the only compositions are (2) and $(1,1)$ so there are a total of 3 parts.]

(8) Fix an integer $k \geq 2$. Call a composition $(a_1, \ldots, a_n)$ of k **doubly even** if the number of a_i which are even is also even (i.e., there could be no even a_i, or 2 of them, or 4, etc.). Show that the number of doubly even compositions of k is 2^{k-2}.

For example, if $k = 4$, then here are the 4 doubly even compositions of 4:

$$(2,2), \qquad (3,1), \qquad (1,3), \qquad (1,1,1,1).$$

(9) Let $F(n)$ be the number of set partitions of $[n]$ such that every block has size ≥ 2. Prove that

$$B(n) = F(n) + F(n+1),$$

where $B(n)$ is the nth Bell number.