Finite Coxeter Groups

(W,S) $\longleftrightarrow$ Γ Coxeter graph

Coxeter group

vertices $= S$

between s & t $(s \neq t)$ draw $m(s,t)-2$ edges.

Represent by $\underset{s}{\cdot}\overset{m(s,t)}{\text{———}}\underset{t}{\cdot}$ if $m(s,t) > 3$

$\underset{s}{\cdot}\text{——}\underset{t}{\cdot}$ if $m(s,t) = 3$

(W,S) is irreducible if Γ is connected

direct product of groups $\longleftrightarrow$ disjoint union of graphs

+ direct sums of geometric representations

$G \subset GL(V)$ subgroup acts *irreducibly* on V if the only G-invariant subspaces of V are 0 and V (V is an *irreducible representation* of G)

$g \in GL(V)$ is a *reflection* if
- g has finite order
- $\mathrm{im}(g-1)$ is 1-dim'l

Lemma. $G \subset GL_n(\mathbb{R})$ act irreducibly on $\mathbb{R}^n$. Assume G contains a reflection.

$\Rightarrow \exists$ unique bilinear form (up to scalar multiple) which is G-invariant ($B(v,w) = B(gv, gw)$ $\forall g \in G$)

Furthermore, up to sign, this bilinear is symmetric + positive definite.

Pf. If B is nonzero G-invariant bilinear form,

$\rightsquigarrow$ $\mathbb{R}^n \longrightarrow (\mathbb{R}^n)^*$ is a G-linear map

$v \longrightarrow B(v,-)$ By irreducibility, this is an isomorphism.

Let B' be another G-invariant bilinear form. For any $v \in \mathbb{R}^n$, $\exists v' \in \mathbb{R}^n$ s.t. $B(v,-) = B'(v',-)$, if we set $\varphi(v) = v'$, then $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ is G-linear.

Let $s \in G$ be a reflection, so $\dim \text{Image}(1-s)=1$, let $(1-s)v$ span it.

$\varphi(1-s)v = (1-s)\varphi v = \alpha(1-s)v$ for some $\alpha \in \mathbb{R}$.

α is eigenvalue for φ. $\varphi - \alpha I$ is also G-linear, its kernel is a nonzero G-invariant subspace. Irreducibility $\Rightarrow \ker(\varphi - \alpha I) = \mathbb{R}^n$

$\Rightarrow B' = \alpha B$.

Consider $B(v,w) = \sum_{g \in G} gv \cdot gw$. symmetric, G-linear

If $v \neq 0$, then $gv \cdot gv > 0 \ \forall g \in G, \Rightarrow B(v,v) > 0$.

$\Rightarrow B$ is pos. definite. $\square$

Thm (Maschke) U vector space / field of char. 0

$G \subset GL(U)$ finite subgroup. If $U' \subseteq U$ is G-invariant, then $\exists$ Ginvariant subspace U'' which is complement to U', i.e.,

$U' + U'' = U$ & $U' \cap U'' = 0$.

Pf. Pick any complementary subspace X to U'. Let $p: U \to U'$ be projection w/ kernel X. Then $\frac{1}{|G|}\sum_{g \in G} gp$ is also a projection which is G-linear. Let U'' be its kernel. $\square$

Lemma. Let (W,S) be Coxeter group. Assume Γ connected & W is finite. Then W acts irreducibly on its geometric representation V.

Pf. Pick $V' \subseteq V$ W-invariant subspace, define $S' = \{ s \in S \mid \alpha_s \in V' \}$.

If $v \in V'$, $t \notin S'$ then $\sigma_t(v) - v = -2B_W(v, \alpha_t)\alpha_t \in V'$

$\Rightarrow B_W(v, \alpha_t) = 0$. $\Rightarrow$ no edges between S' and $S \setminus S'$ in Γ.

Γ connected $\Rightarrow$ $S' = S$ or $S' = \emptyset$.

Case 1. $S' = S \Rightarrow V = V'$ ✓

Case 2. $S' = \emptyset \Rightarrow V' \subseteq \ker B_W$. We will show $\ker B_W = 0$.

Since $\ker B_W$ is W-invariant, Maschke gives W-inv't complement V''.

If $V'' = V$, then $\ker B_W = 0$ ✓

Otherwise V'' is a proper subspace, $\Rightarrow V'' \subseteq \ker B_W$.

Since $V'' \cap \ker B_W = 0 \Rightarrow V'' = 0 \Rightarrow \ker B_W = V \rightarrow\leftarrow$ $\square$

Rmk. Fails if W infinite: If W = infinite dihedral group, $S = \{s, t\}$.
we saw that $\mathrm{span}(\alpha_s + \alpha_t)$ is a W-invariant subspace of V.

Prop. W is finite $\Longleftrightarrow$ B_W is positive definite.

Pf. Assume W finite, w/ geometric representation $W \subseteq GL(V)$.
Suffices to assume Γ is connected. $\Rightarrow W \curvearrowright V$ is irreducible.
$\Rightarrow B_W$ is either positive definite or negative definite.
$B_W(\alpha_s, \alpha_s) = 1 \ \forall s \in S \Rightarrow$ positive definite.

Suppose B_W is positive definite. By picking orthonormal basis, group preserving B_W is $\{A \mid A^T A = I\}$, which is compact.
(closed & bounded in $\mathbb{R}^{n^2}$)

$W \subseteq GL_n(\mathbb{R})$ is discrete (by previous lecture)

discrete + compact $\Rightarrow$ finite $\square$

Thm. Any finite subgroup $G \subseteq GL_n(\mathbb{R})$ generated by reflections is isomorphic to a Coxeter group.

Pf. First, $\mathbb{R}^n$ has symmetric positive definite G-invariant bilinear form
$(v, w) = \sum_{g \in G} gv \cdot gw$. Change basis so that this becomes dot product.

Given hyperplane $H \subseteq \mathbb{R}^n$, let s_H be reflection which fixes H and negates a normal vector. $\mathcal{H}_G = \{H \mid s_H \in G\}$

$\mathbb{R}^n \setminus \bigcup_{H \in \mathcal{H}_G} H$ has finitely many components (called <u>chambers</u>)

Let C be chamber, $\overline{C}$ = closure of C.

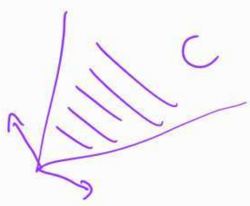
$S = \{s_H \mid H \text{ bounds } \overline{C}\}$. Let G' = group generated by S.

Claim: Every G'-orbit of a vector in $\mathbb{R}^n$ has nonempty intersection w/ $\overline{C}$.

Pick $a \in C$, $v \in \mathbb{R}^n$. $G' \cdot v$ is finite $\Rightarrow \exists$ element $v' \in G' \cdot v$ whose distance to a is minimized. If $v' \notin \overline{C}$, $\exists H$ bounding $\overline{C}$ s.t. a, v' are on separate sides of H. $s_H(v')$ is now closer to $a \rightarrow\!\!\leftarrow$
$\Rightarrow v' \in \overline{C}$ □

Claim $\Rightarrow G'$ applied to any chamber will contain C
$\Rightarrow$ For any $s_H \in G$, $\exists g \in G'$ s.t. $g s_H g^{-1} = s_{gH} \in S$
$\Rightarrow s_H \in G' \Rightarrow G'$ contains all reflections of $G \Rightarrow G' = G$.

For each $s \in S$, let α_s be unit normal vector to $\ker(s-1)$ pointing in direction of C



$(\alpha_s, a) > 0$ for all $a \in C$

& $(\alpha_s, \alpha_t) \leq 0$ for $s \neq t$. [Reduce to considering $\mathrm{span}(\alpha_s, \alpha_t)$]

Claim: α_s are linearly independent.

Pf. Suppose we have $\sum_{s \in S} c_s \alpha_s = 0$. Let $v = \sum_{\substack{s \in S \\ c_s > 0}} c_s \alpha_s$

- If $v = 0$, then no c_s are positive. If not, $0 = (a, v) = \sum_{\substack{s \in S \\ c_s > 0}} c_s (\alpha_s, a)$. ($a \in C$)

$$\Rightarrow \overset{a\in C}{0} = (a, \sum c_s \alpha_s) = \sum \underbrace{c_s}_{\leq 0} \underbrace{(a, \alpha_s)}_{\text{pos.}} \Rightarrow \text{ all } c_s = 0 \text{, dependency is trivial.}$$

- If $v \neq 0$, so

$$0 < (v,v) = \sum_{\substack{s\in S \\ c_s>0}} \sum_{\substack{t\in S \\ c_t<0}} \underbrace{-c_s c_t}_{>0} \underbrace{(\alpha_s, \alpha_t)}_{\leq 0} \quad \rightarrow\leftarrow \quad \square$$

For $s,t \in S$, let $m(s,t)=$ order of st. By restricting to span$\{\alpha_s, \alpha_t\}$ we can conclude that $(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m(s,t)}\right)$.

We have surjective homomorphism $W \to G$ where W is Coxeter group defined by S, m. V is geom. representation of W

$\Rightarrow \quad W \longrightarrow GL(V)$ is injective $\Rightarrow W \cong G$. $\square$