Poincaré series (W,S) Coxeter group.

Def. Given $U \subseteq W$, define $U(t) = \sum_{w \in U} t^{\ell(w)} = \sum_{n \geq 0} |\{w \in U \mid \ell(w) = n\}| t^n$

$\Longrightarrow$ for any $I \subseteq S$, $W(t) = W_I(t)\, W^I(t)$

$[\forall w \in W, \exists$ unique $u \in W^I, v \in W_I$ s.t. $w = uv$ & $\ell(w) = \ell(u) + \ell(v)]$

Lemma. If W finite, $\exists$ unique element w_0 of maximal possible length.

$w_0(\Phi^+) = \Phi^-$.

Pf. First, # roots is finite. Let w_0 be an element of maximal possible length. $\Rightarrow \forall s \in S,\ \ell(w_0 s) < \ell(w_0) \Rightarrow w(\alpha_s) < 0\ \forall s \in S \Leftrightarrow w_0(\Phi^+) = \Phi^-$.

$\Longrightarrow \ell(w_0) = \#$ pos. roots.

If $\ell(w') = \ell(w_0) \Rightarrow w'(\Phi^+) = \Phi^- \Rightarrow w' w_0(\Phi^+) = \Phi^+ \Rightarrow \ell(w' w_0) = 0$

$\Rightarrow w' w_0 = 1$. [If we take $w' = w_0 \Rightarrow w_0 = w_0^{-1}$]

$\Rightarrow w' = w_0^{-1} = w_0$. $\square$

Given $w \in W$, (right) descent set is $D_R(w) = \{s \in S \mid \ell(ws) < \ell(w)\}$.

$W^I = \{w \in W \mid D_R(w) \subseteq S \setminus I\}$.

Prop. $$\sum_{I \subseteq S} (-1)^{|I|} W^I(t) = \begin{cases} t^{\ell(w_0)} & \text{if } W \text{ finite.} \\ 0 & \text{else} \end{cases}$$

Pf. For $w \in W$, contribution to sum is
$$t^{\ell(w)} \sum_{I \subseteq S \setminus D_R(w)} (-1)^{|I|} = \begin{cases} t^{\ell(w)} & \text{if } S = D_R(w) \\ 0 & \text{else} \end{cases}$$

$D_R(w) = S \Leftrightarrow w(\alpha_s) < 0\ \forall s \in S. \Leftrightarrow w(\Phi^+) = \Phi^-$.

If W is infinite, no such element exists: if it did, its ℓ is # pos. roots, but ℓ is unbounded $\Rightarrow \sum_{I \subseteq S} (-1)^{|I|} W^I(t) = 0$

If W is finite, $\exists$ unique w_0 w/ $D_R(w_0) = S \Rightarrow \sum_{I \subseteq S} (-1)^{|I|} W^I(t) = t^{\ell(w_0)}$ $\square$

Note: $W^S(t) = 1$, subtract it from identity:

Define $f(t) = \begin{cases} t^{\ell(w_0)} - (-1)^{|S|} & \text{if } W \text{ finite} \\ -(-1)^{|S|} & \text{if } W \text{ infinite} \end{cases}$

$$\sum_{I \subsetneq S} (-1)^{|I|} W^I(t) = f(t).$$ Divide by $W(t)$ (possible b/c constant term of $w(t)$ is 1)

$$\sum_{I \subsetneq S} (-1)^{|I|} \frac{1}{W_I(t)} = \frac{f(t)}{W(t)}$$

By induction on $|S|$, these can be considered known.

Can determine $W(t)$ from this data if W infinite, or if we know $\ell(w_0)$ if W finite.

Ex. If $|S| = 0$, then $W(t) = 1$

If $|S| = 1$, $W \cong \mathbb{Z}/2$, $W(t) = 1+t$.

$$\underbrace{\sum_{I \subsetneq S} (-1)^{|I|} \frac{1}{W_I(t)}}_{1} = \frac{f(t)}{W(t)} \sim \frac{t+1}{t+1}$$

Ex. $W = \mathfrak{S}_n$. Represent W in 1-line notation: $w(1)\, w(2) \cdots w(n)$

can put $n+1$ in any place

inversions of result is $\ell(w) + j$

where j values appear after $n+1$. j can be anything between $0, 1, \ldots, n$.

$$\Rightarrow \mathfrak{S}_{n+1}(t) = \mathfrak{S}_n(t)(1 + t + \cdots + t^n) = \mathfrak{S}_n(t) \frac{1-t^{n+1}}{1-t}$$

$$\Rightarrow \mathfrak{S}_n(t) = \frac{(1-t)(1-t^2)\cdots(1-t^n)}{(1-t)^n}$$