Thm. Let $w = s_1 \cdots s_r$, $s_i \in S$. Pick $t \in T$ (reflection) s.t. $l(wt) < l(w)$.
$\Rightarrow \exists\, 1 \le j \le r$ s.t. $wt = s_1 \cdots \hat{s}_j \cdots s_r$. If $s_1 \cdots s_r$ reduced, then choice of j is unique.

Pf. Write $t = s_\alpha$, $\alpha \in \Phi^+$. $l(wt) < l(w) \Rightarrow w(\alpha) < 0$.
Hence $\exists j$ s.t. $s_j s_{j+1} \cdots s_r(\alpha) < 0$, but, $s_{j+1} \cdots s_r(\alpha) > 0$.
Since s_j preserves $\Phi^+ \setminus \{\alpha_{s_j}\}$, we know that $s_{j+1} \cdots s_r(\alpha) = \alpha_{s_j}$.
$\Rightarrow t = (s_{j+1} \cdots s_r)^{-1} s_j (s_{j+1} \cdots s_r) = s_r \cdots s_{j+1} s_j s_{j+1} \cdots s_r$.
$\Rightarrow wt = (s_1 \cdots s_r)(s_r \cdots s_{j+1}) s_j (s_{j+1} \cdots s_r) = s_1 \cdots s_{j-1} s_{j+1} \cdots s_r = s_1 \cdots \hat{s}_j \cdots s_r$.
Suppose $\exists\, i < j$ s.t. $wt = \underbrace{s_1 \cdots \hat{s}_i \cdots s_r = s_1 \cdots \hat{s}_j \cdots s_r}$
$\Rightarrow s_{i+1} \cdots s_j = s_i \cdots s_{j-1} \Rightarrow s_i \cdots s_j = s_{i+1} \cdots s_{j-1}$
$\Rightarrow w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$, so original expression not reduced. $\square$

Cor. If $w = s_1 \cdots s_r$ not reduced, then $\exists\, i < j$ s.t. $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$.
In particular, we can delete an even # of the s_i to get reduced expression.

Pf. Since $l(w) < r$, $\exists j$ s.t. $l(s_1 \cdots s_j) < j$. Let $w' = s_1 \cdots s_{j-1}$.
By Thm ($t = s_j$), $\exists\, i \le j-1$ s.t. $w' s_j = s_1 \cdots \hat{s}_i \cdots s_{j-1}$.
$\Rightarrow w = (s_1 \cdots s_j) s_{j+1} \cdots s_r = (s_1 \cdots \hat{s}_i \cdots s_{j-1}) s_{j+1} \cdots s_r$. $\square$

Let $I \subseteq S$, define $W^I = \{ w \in W \mid l(ws) > l(w) \text{ for all } s \in I \}$.

Prop. Given $w \in W$, $\exists$ unique $u \in W^I$ & $v \in W_I$ s.t. $w = uv$.
Furthermore, $l(w) = l(u) + l(v)$. Moreover, u is unique element of minimal possible length in coset wW_I.

Pf Pick $u \in wW_I$ of minimal possible length, and $v = u^{-1}w \in W_I$.
- Then $u \in W^I$: Pick $s \in I$: then $us \in W_I$, so by choice of u, $l(us) > l(u)$
- Pick reduced expressions $u = s_1 \cdots s_p$ & $v = s_1' \cdots s_q'$. ($\Rightarrow s_i' \in I$)

$w = (s_1 \cdots s_p)(s_1' \cdots s_q')$. Suppose not reduced: then can delete 2 elements to get shorter expression for w. If one of these is an s_i, then we get another coset representative for w w/ smaller length than $u \rightarrow\leftarrow$

So both elements must be $s_i', s_j' \Rightarrow v = s_1' \cdots \hat{s_i'} \cdots \hat{s_j'} \cdots s_q \rightarrow\leftarrow$

$\Rightarrow \ell(w) = \ell(u) + \ell(v)$.

- Suppose $\exists\, u' \in wW_I$ s.t. $\ell(u') = \ell(u)$. Then $u' = uv'$ for some $v' \in W_I$.

  Repeating previous argument, $\ell(u') = \ell(u) + \ell(v') \Rightarrow \ell(v') = 0 \Rightarrow v' = 1 \Rightarrow u = u'$.

- Suppose $\exists\, u' \in W^I$ s.t. $u' \in wW_I$. & $u' \neq u$. $\Rightarrow \underbrace{\ell(u') - \ell(u)}_{r} > 0$

  $\Rightarrow u' = u s_1 \cdots s_r$ for $s_i \in I$. $\Rightarrow \ell(u's_r) < \ell(u') \rightarrow\leftarrow$. $\square$

Elements of W^I are minimal length coset representatives

Ex. $W = \mathfrak{S}_n$ Pick $i < n$ $I = \{s_1, \ldots \hat{s_i}, \ldots s_{n-1}\}$

$W_I \cong \mathfrak{S}_i \times \mathfrak{S}_{n-i}$. $W^I = \left\{ w \in \mathfrak{S}_n \;\middle|\; \begin{array}{l} w(1) < w(2) < \cdots < w(i) \\ w(i+1) < w(i+2) < \cdots < w(n) \end{array} \right\}$

$W^I \cong$ i-element subsets of $\{1, \ldots, n\}$

$|W^I| = \binom{n}{i} = \dfrac{n!}{i!(n-i)!}$

Bruhat order. Given $v, w \in W$, write $v \rightarrow w$ if $\ell(w) > \ell(v)$ & $v^{-1}w \in T$ (i.e., $\exists\, t \in T$ s.t. $vt = w$)

Note: $\rightarrow$ is antisymmetric, we let $\leq$ (Bruhat order) be the partial order generated by $\rightarrow$, i.e., $w' \leq w$ if $\exists\, w_0, \ldots w_n$ s.t.

$w' = w_0 \rightarrow w_1 \rightarrow \cdots \rightarrow w_n = w$

Prop. Let $w' \leq w$, pick $s \in S$. Then at least one of the following holds:

① $w's \leq w$, ② $w's \leq ws$

Pf. First suppose $w' \rightarrow w$! So $\exists\, t \in T$ s.t. $w = w't$ & $\ell(w) > \ell(w')$.

If $s = t$, then ① holds, so suppose $s \neq t$.

Case1: $\ell(w's) = \ell(w')-1$. Then $w's \to w' \to w \Rightarrow w's \leq w \Rightarrow$ (1)

Case2. $\ell(w's) = \ell(w')+1$. Pick reduced expression $w' = s_1 \cdots s_r$. Let $s_{r+1} = s$.

Then $s_1 \cdots s_{r+1} = w's$ reduced. Set $t' = sts$, so $ws = w'st'$.

Claim: $\ell(w's) < \ell(ws)$. If not, then $\ell(w'st') < \ell(w's) \Rightarrow$

$\exists\, j \leq r+1$ s.t. $s_1 \cdots \hat{s_j} \cdots s_{r+1} = w'st' = ws$ is reduced.

Since $s \neq t$, we have $j \leq r \Rightarrow s_1 \cdots \hat{s_j} \cdots s_r = w \Rightarrow \ell(w) \leq r-1$

$\Rightarrow \ell(w') < \ell(w) \to\leftarrow$

$\Rightarrow w's \to ws \Rightarrow$ (2)

General case: $\exists\, w_0, \ldots, w_n$ s.t. $w' = w_0 \to w_1 \to \cdots \to w_n = w$.

Induction on n: base $n=0$ (i.e., $w'=w$) ✓

Suppose $n>0$: From what we've done, using $w' \to w_1$ we know that either $w's \leq w_1$ or $w's \leq w_1 s$.

Case1 $w's \leq w_1$. Since $w_1 \leq w \Rightarrow w's \leq w_1 \leq w. \Rightarrow$ (1)

Case2. $w's \leq w_1 s$. Use induction on $w_1 \to w_2 \to \cdots \to w_n = w$

$\Rightarrow w_1 s \leq w$ or $w_1 s \leq ws$.

Case 2a. $w_1 s \leq w \Rightarrow w's \leq w_1 s \leq w \Rightarrow$ (1)

Case 2b. $w_1 s \leq ws \Rightarrow w's \leq w_1 s \leq ws \Rightarrow$ (2). □

Given an expression $s_1 \cdots s_r$, a subword is $s_{i_1} \cdots s_{i_p}$ where $1 \leq i_1 < i_2 < \cdots < i_p \leq r$

Thm. Pick $w \in W$, $w = s_1 \cdots s_r$ reduced. Then $v \leq w \Leftrightarrow$ some subword of $s_1 \cdots s_r$ multiplies to v. $\Rightarrow$ # of v s.t. $v \leq w$ is finite.

Pf. First suppose $v \leq w$, so $v = w_0 \to w_1 \to \cdots \to w_n = w$. We show that some subword of $s_1 \cdots s_r$ gives v by induction on n.

Base case $n=0$: $v = w$ ✓

Assume $n \geq 1$. Then $w_{n-1} \to w \Rightarrow w_{n-1}^{-1} w \in T$ & $\ell(w_{n-1}) < \ell(w)$.

$\exists\, j \le r$ s.t. $w_{n-1} = s_1 \cdots \hat{s}_j \cdots s_r \Rightarrow$ Some subword of x is reduced expression of w_{n-1}

By induction on n, using $v = w_0 \to w_1 \to \cdots \to w_{n-1}$, we know some subword of x gives v, this is also subword of $s_1 \cdots s_r$.

Prove other implication by induction on $\ell(w) = r$.

Base case $r = 0$: $w = 1$

Assume $r \ge 1$. $w = s_1 \cdots s_r$, $v = s_{i_1} \cdots s_{i_p}$.

Case 1. $i_p < r$. Then v is subword of $s_1 \cdots s_{r-1}$. By induction

$v \le ws_r$, but $ws_r \to w \Rightarrow v \le ws_r \le w$.

Case 2. $i_p = r$. $s_{i_1} \cdots s_{i_{p-1}}$ is subword of $s_1 \cdots s_{r-1}$

By induction $vs_r \le ws_r$. By previous result set

$$\underbrace{vs_r s_r \le ws_r}_{v \le ws_r \le w \checkmark} \quad \text{or} \quad \underbrace{vs_r s_r \le ws_r s_r}_{v \le w \checkmark}$$

$\square$

Ex. Dihedral group of order $2m$ w/ $S = \{s, t\}$.

$W =$

1	
s	t
st	ts
sts	tst
$\vdots$	

alt. products of length $m-1$

$tst\cdots = sts\cdots$ length m

$\Rightarrow$ $v \le w \iff \ell(v) < \ell(w)$

Also valid if $m = \infty$

Cor. For $I \subseteq S$, restriction of $\le$ on W to W_I is the Bruhat order of W_I.

Pf. Pick $v, w \in W_I$. Every reduced expression for w only uses $s \in I$.

Fix one. Then "v is a subword of this expression" does not depend on ambient Coxeter group. $\square$