Kazhdan-Lusztig polynomials

Define $\iota : \mathbb{Z}[q^{\pm 1/2}] \to \mathbb{Z}[q^{\pm 1/2}]$, extend to $\mathcal{H}$ by $\iota(T_w) = T_{w^{-1}}^{-1}$
$\iota(q^{1/2}) = q^{-1/2}$

Lemma. ι is a ring homomorphism; in particular, $\iota^2 = 1$

Pf. Suffices to show $\iota(T_s)\iota(T_w) = \iota(T_s T_w)$. $\forall s \in S$, $w \in W$.

Case 1: $sw > w$:

$$\iota(T_s T_w) = \iota(T_{sw}) = T_{w^{-1}s}^{-1} = (T_{w^{-1}} T_s)^{-1} = T_s^{-1} T_{w^{-1}}^{-1} = \iota(T_s)\iota(T_w).$$

Case 2: $sw < w$.

$$\iota(T_s T_w) = \iota((q-1)T_w + q T_{sw}) = (q^{-1}-1) T_{w^{-1}}^{-1} + q^{-1} T_{w^{-1}s}^{-1}$$

$$\iota(T_s) = T_s^{-1} = q^{-1}(T_s - (q-1))$$

$$\Rightarrow \iota(T_s)\iota(T_w) = q^{-1} T_s T_{w^{-1}}^{-1} + (q^{-1}-1) T_{w^{-1}}^{-1}$$

Since $w^{-1} > w^{-1}s$, so $T_{w^{-1}s} T_s T_{w^{-1}}^{-1} = T_{w^{-1}} T_{w^{-1}}^{-1} = 1$

$$\Rightarrow T_s T_{w^{-1}}^{-1} = T_{w^{-1}s}^{-1} \Rightarrow \iota(T_s)\iota(T_w) = \iota(T_s T_w).$$ $\square$

Cor. For all $x \le w$,

$$\sum_{x \le y \le w} \varepsilon_x \varepsilon_y R_{x,y}(q) R_{y,w}(q) = \delta_{x,w}$$

Pf. Apply ι to $T_{w^{-1}}^{-1} = \sum_{x \le w} q^{-\ell(x)} R_{x,w}(q^{-1}) T_x$

to get $T_w = \sum_{y \le w} q^{\ell(y)} R_{y,w}(q) T_{y^{-1}}^{-1}$

$$\Downarrow$$

$$T_{y^{-1}}^{-1} = \varepsilon_y q^{-\ell(y)} \sum_{x \le y} \varepsilon_x R_{x,y}(q) T_x$$

$$T_w = \sum_{y \le w} \cancel{q^{\ell(y)}} R_{y,w}(q)\, \varepsilon_y \cancel{q^{-\ell(y)}} \sum_{x \le y} \varepsilon_x R_{x,y}(q)\, T_x$$

$$= \sum_{x \le y \le w} \varepsilon_y \varepsilon_x R_{y,w}(q) R_{x,y}(q) T_x \quad \text{[take coeff of } T_x \text{]}$$

□

Thm. For each $w \in W$, $\exists$ unique $C_w \in \mathcal{H}$ s.t.

(1) $\iota(C_w) = C_w$

(2) $\exists$ polynomials $P_{x,w}(q)$ for $x \le w$ s.t. $P_{w,w}(q) = 1$ &

$\deg P_{x,w}(q) \le \frac{1}{2}(\ell(w) - \ell(x) - 1)$ for $x < w$ &

$$C_w = \varepsilon_w q^{\ell(w)/2} \sum_{x \le w} \varepsilon_x q^{-\ell(x)} P_{x,w}(q^{-1}) T_x .$$

$P_{x,w}(q)$ are Kazhdan-Lusztig polynomials

Pf. Pick $w \in W$. Consider element of the form

$$C'_w = \varepsilon_w q^{\ell(w)/2} \sum_{y \le w} \varepsilon_y q^{-\ell(y)} P'_{y,w}(q^{-1}) T_y .$$

$$\iota(C'_w) = \varepsilon_w q^{-\ell(w)/2} \sum_{y \le w} \cancel{\varepsilon_y} \cancel{q^{\ell(y)}} P'_{y,w}(q) \sum_{x \le y} \cancel{\varepsilon_y} \cancel{q^{-\ell(y)}} \varepsilon_x R_{x,y}(q) T_x$$

$$= \varepsilon_w q^{-\ell(w)/2} \sum_{x \le y \le w} \varepsilon_x P'_{y,w}(q) R_{x,y}(q) T_x$$

For $x \le w$, coefficient of T_x in $\varepsilon_x \varepsilon_w q^{\ell(x)/2}(C'_w - \iota(C'_w))$ is

$$\underbrace{q^{\frac{\ell(w)-\ell(x)}{2}} P'_{x,w}(q^{-1})}_{1} - \underbrace{q^{\frac{-\ell(w)+\ell(x)}{2}} P'_{x,w}(q)}_{2} - \underbrace{\sum_{x < y \le w} q^{\frac{-\ell(w)+\ell(x)}{2}} P'_{y,w}(q) R_{x,y}(q)}_{3}$$

want all these expressions 0 s.t. $P'_{ww} = 1$ & $\deg P'_{x,w} \le \frac{1}{2}(\ell(w) - \ell(x) - 1)$

Induction on $\ell(w)-\ell(x)$. Base case: set $P'_{w,w}(q)=1$ $\forall w \in W$.

Pick $x<w$. ① is a polynomial in $q^{1/2}$
② is a polynomial in $q^{-1/2}$

$\Rightarrow$ ① & ② can't cancel each other $\Rightarrow$ at most solution for $P'_{x,w}$ exists

let $\alpha =$ ③ Solution for $P'_{x,w}$ exists s.t. red $=0$

$\Longleftrightarrow \iota(\alpha) = -\alpha$.

$$\iota(\alpha) = \sum_{x<z\le w} R_{x,z}(q^{-1})\, q^{\frac{\ell(w)-\ell(x)}{2}} P'_{z,w}(q^{-1})$$

$$= \sum_{x<z\le w} \varepsilon_x \varepsilon_z q^{\ell(x)-\ell(z)} R_{x,z}(q)\, q^{\frac{\cancel{\ell(w)}-\ell(x)}{2}} q^{\frac{-\cancel{\ell(w)}+\ell(z)}{2}} \sum_{z\le y\le w} q^{\frac{-\ell(w)+\ell(z)}{2}} P'_{y,w}(q) R_{z,y}(q)$$

$$= \sum_{x<z\le y\le w} q^{\frac{\ell(x)-\ell(w)}{2}} P'_{y,w}(q)\, \varepsilon_x \varepsilon_z R_{x,z}(q) R_{z,y}(q)$$

$$= \sum_{x<y\le w} q^{\frac{\ell(x)-\ell(w)}{2}} P'_{y,w}(q) \sum_{x<z\le y} \varepsilon_x \varepsilon_z R_{x,z}(q) R_{z,y}(q)$$

$$= \sum_{x<y\le w} q^{\frac{\ell(x)-\ell(w)}{2}} P'_{y,w}(q) \left(-R_{x,y}(q)\right) = -\alpha. \qquad \square$$