R-polynomials

For $s \in S$, $T_s^2 = (q-1)T_s + q \implies T_s(T_s + 1 - q) = q$

$\implies T_s^{-1} = q^{-1}(T_s - (q-1))$

$\implies T_w$ invertible for all $w \in W$.

Set $\varepsilon_w = (-1)^{\ell(w)}$

Thm. For $x \leq w$, $\exists$ polynomials $R_{x,w}(q)$ of degree $\ell(w) - \ell(x)$ s.t.

$R_{w,w}(q) = 1$, and
$$T_{w^{-1}}^{-1} = \varepsilon_w q^{-\ell(w)} \sum_{x \leq w} \varepsilon_x R_{x,w}(q) T_x$$

Furthermore, these polynomials are nonzero.

Pf. Induction on $\ell(w)$. Base case clear.

If $\ell(w) > 0$, write $w = sv$ for $s \in S$, $\ell(v) = \ell(w) - 1$.

$$T_{w^{-1}}^{-1} = T_s^{-1} T_{v^{-1}}^{-1}$$
$$= q^{-1}(T_s - (q-1)) \varepsilon_v q^{-\ell(v)} \sum_{y \leq v} \varepsilon_y R_{y,v}(q) T_y$$
$$= \varepsilon_v q^{-\ell(w)} \sum_{y \leq v} \varepsilon_y R_{y,v}(q) T_s T_y - \varepsilon_v q^{-\ell(w)} \sum_{y \leq v} \varepsilon_y (q-1) R_{y,v}(q) T_y$$
$$= \varepsilon_v q^{-\ell(w)} \sum_{\substack{y \leq v \\ \ell(sy) > \ell(y)}} \varepsilon_y R_{y,v}(q) T_{sy} + \varepsilon_v q^{-\ell(w)} \sum_{\substack{y \leq v \\ \ell(sy) < \ell(y)}} \varepsilon_y R_{y,v}(q) ((q-1)T_y + q T_{sy})$$
$$- \varepsilon_v q^{-\ell(w)} \sum_{y \leq v} \varepsilon_y (q-1) R_{y,v}(q) T_y$$
$$= \varepsilon_v q^{-\ell(w)} \sum_{\substack{y \leq v \\ \ell(sy) > \ell(y)}} \varepsilon_y R_{y,v}(q) T_{sy} + \varepsilon_v q^{-\ell(w)} \sum_{\substack{y \leq v \\ \ell(sy) < \ell(y)}} \varepsilon_y R_{y,v}(q)\, q\, T_{sy}$$

$$- \varepsilon_v q^{-\ell(w)} \sum_{\substack{y \le v \\ \ell(sy) > \ell(y)}} \varepsilon_y (q-1) R_{y,v}(q) \underline{T_y}$$ all $\le w$

Pick $x \le w$ and consider its coefficient in this expression.

Case 1. $x > sx$. Then T_x only appears in the first sum w/ $sy = x$

so we let $R_{x,w}(q) = R_{y,v}(q) = R_{sx,sw}(q)$ (nonzero)

$$\deg R_{x,w}(q) = \ell(sw) - \ell(sx) = \ell(w) - 1 - (\ell(x) - 1) = \ell(w) - \ell(x).$$

Case 2. $x < sx$. Then T_x not in first sum, but does appear in other two, we take $R_{x,w}(q) = q R_{sx,sw}(q) + (q-1) R_{x,sw}(q)$

$$\deg(q R_{sx,sw}(q)) = 1 + \ell(sw) - \ell(sx) = 1 + (\ell(w) - 1) - (\ell(x) + 1)$$
$$= \ell(w) - \ell(x) - 1$$

$$\deg((q-1) R_{x,sw}(q)) = 1 + \ell(sw) - \ell(x) = 1 + \ell(w) - 1 - \ell(x) = \ell(w) - \ell(x)$$
$$\Rightarrow \deg R_{x,w}(q) = \ell(w) - \ell(x). \qquad \square$$

$R_{x,w}(q)$ are <u>R-polynomials</u>.

Cor. (Assume $R_{a,b}(q) = 0$ if $a \not\le b$).
Pick $s \in S$, so that $sw < w$. For $x \le w$, have

$$R_{x,w}(q) = \begin{cases} R_{sx,sw}(q) & \text{if } x > sx. \\ q R_{sx,sw}(q) + (q-1) R_{x,sw}(q) & \text{if } x < sx \end{cases}$$

Cor. If $x \le w$, $R_{x,w}(q^{-1}) = \varepsilon_x \varepsilon_w q^{\ell(x) - \ell(w)} R_{x,w}(q)$

Pf. Induction on $\ell(w)$. Base case: $1 = 1$ ✓

Else pick $s \in S$ s.t. $sw < w$.

Case 1. $x > sx$: $R_{x,w}(q^{-1}) = R_{sx,sw}(q^{-1}) = \varepsilon_{sx} \varepsilon_{sw} q^{\ell(sx) - \ell(sw)} R_{sx,sw}(q)$

$$= \varepsilon_x \varepsilon_w q^{\ell(x)-\ell(w)} R_{x,w}(q).$$

<u>Case2.</u> $x < sx$. $R_{x,w}(q^{-1}) = q^{-1} R_{sx,sw}(q^{-1}) + (q^{-1}-1) R_{x,sw}(q^{-1})$

$$= q^{-1} \varepsilon_{sx}\varepsilon_{sw} q^{\ell(sx)-\ell(sw)} R_{sx,sw}(q) + (q^{-1}-1)\varepsilon_x \varepsilon_{sw} q^{\ell(x)-\ell(sw)} R_{x,sw}(q)$$

$\ell(sx) = \ell(x)+1$, $\ell(sw) = \ell(w)-1$

$$= \varepsilon_x \varepsilon_w q^{\ell(x)-\ell(w)+1} R_{sx,sw}(q) - (1-q)\varepsilon_x \varepsilon_w q^{\ell(x)-\ell(w)} R_{x,sw}(q)$$

$$= \varepsilon_x \varepsilon_w q^{\ell(x)-\ell(w)} \left[q R_{sx,sw}(q) - (1-q) R_{x,sw}(q) \right]$$

$$= \varepsilon_x \varepsilon_w q^{\ell(x)-\ell(w)} R_{x,w}(q). \qquad \square$$

<u>Cor.</u> For $w \in W$,

$$T_{w^{-1}}^{-1} = \sum_{x \le w} q^{-\ell(x)} R_{x,w}(q^{-1}) T_x.$$