Toroidal Coxeter complex

$W_a \cong T_{L^\vee} \rtimes W$, V = geom. rep. of W

V is triangulated by W_a-translates of $A = \overline{A}_0$ ← fund. alcove

Each $s \in S_a$ gives facet of A:

$\{v \in A \mid (v, \alpha_s) = 0\}$ for $s \in S$, $\{v \in A \mid (v, \tilde{\alpha}) = 1\}$ for $s_0 = S_a \setminus S$

Given $I \subseteq S_a$, define A_I to be intersection of these facets.

A simplex $\Rightarrow$ $A_I \neq \emptyset$ if $I \neq S_a$

Consider quotient $V/L^\vee$ ← n-dim torus $\cong (S^1)^{\times n}$

$\circlearrowleft W$ inherits finite triangulation

toroidal Coxeter complex

$\rightsquigarrow$ chain complex $\mathbb{F}_\cdot$ which computes $H_\cdot(V/L^\vee; \mathbb{R})$

where $\mathbb{F}_i$ = $\mathbb{R}$-vector space w/ basis $\{W\text{-translates of } A_I \mid i = n - |I|\}$

$(i = 0, \ldots, n)$

$$H_i(V/L^\vee; \mathbb{R}) \cong \bigwedge^i V$$

↖ compatible w/ W-action

Given $I \subsetneq S_a$, we know $(W_a)_I$ is finite.

Let $W(I)$ be image of $(W_a)_I$ under $W_a \to W_a / T_{L^\vee} \cong W$

Note: $(W_a)_I$ finite & $T_{L^\vee} \cong \mathbb{Z}^n$ (no torsion)

$\Rightarrow (W_a)_I \cap T_{L^\vee}$ is trivial $\Rightarrow W(I) \cong (W_a)_I$.

If $s_0 \in I$, $W(I)$ is not parabolic subgroup of W

But $W(I)$ is generated by reflections always

Prop. As class functions on W,

$$\sum_{i=0}^{n} (-1)^i \operatorname{char}(\Lambda^i V) = (-1)^n \sum_{I \subsetneq S_a} (-1)^{|I|} 1\uparrow_{W(I)}^{W}$$

Pf. Triangulation is W-invariant, so each $w \in W$ gives chain map $w: \mathcal{F}_\bullet \to \mathcal{F}_\bullet$ and Hopf trace formula gives

$$\sum_{i=0}^{n} (-1)^i \operatorname{Tr}(w | H_i(V/L^\vee; \mathbb{R})) = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}(w | \mathcal{F}_i)$$

Next, $\mathcal{F}_i$ perm. rep. on i-dim'l faces. In V, there is one orbit for each $I \subseteq S_a$ w/ $|I| = n-i$, and A_I are representatives.

The stabilizer of A_I in W_a is $(W_a)_I$.

$\Rightarrow$ The stabilizer of A_I in $V/L^\vee$ is $W(I)$. $\square$

Thm (Bott). Let $d_1, \ldots, d_n$ be degrees of basic invariants for W. Then

$$W_a(t) = W(t) \prod_{i=1}^{n} \frac{1}{1-t^{d_i - 1}} = \frac{1}{(1-t)^n} \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t^{d_i-1}}$$

$\boxed{A = \operatorname{Sym}(V)}$

Pf. $$\sum_{I \subsetneq S_a} (-1)^{|I|} H_{A^{W(I)}}(t) = \sum_{I \subsetneq S_a} (-1)^{|I|} \sum_{d \geq 0} \langle 1\uparrow_{W(I)}^{W}, \operatorname{char}(A_d) \rangle_W t^d$$

Frobenius reciprocity

$$= (-1)^n \sum_{i=0}^{n} (-1)^i \sum_{d \geq 0} \langle \operatorname{char}(\Lambda^i V), \operatorname{char}(A_d) \rangle_W t^d$$

$$= (-1)^n \sum_{i=0}^{n} (-1)^i \sum_{d \geq 0} \dim(\Lambda^i V^* \otimes A_d)^W t^d \qquad (V^* \cong V)$$

$$= (-1)^n H_{(A\otimes E)^W}(t,-1)$$

$$= (-1)^n \prod_{i=1}^{n} \frac{1-t^{d_i-1}}{1-t^{d_i}}$$

Next, each $W(I)$ is finite reflection group, so

$$H_{A^{W(I)}}(t) \cdot (1-t)^n = \frac{1}{\underset{\text{Hilbert series of Coinvariant ring wrt } W(I)}{\rule{3em}{0.4pt}}} = \frac{1}{W(I)(t)}$$

$$\Rightarrow \frac{(-1)^n}{W_a(t)} = \sum_{I \subsetneq S_a} \frac{(-1)^{|I|}}{W(I)(t)} = \sum_{I \subsetneq S_a} (-1)^{|I|} H_{A^{W(I)}}(t) (1-t)^n$$

$$= (-1)^n (1-t)^n \prod_{i=1}^{n} \frac{1-t^{d_i-1}}{1-t^{d_i}}$$

$$\Rightarrow \quad W_a(t) = \frac{1}{(1-t)^n} \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t^{d_i-1}}. \qquad \square$$

This proof is due to Steinberg.