Alcoves

$$D = \{ f \in V_a^* \mid f(\alpha_s) \geq 0 \quad \text{for all } s \in S_a \}$$

$$E = \{ f \in V_a^* \mid f(\delta) = 1 \} \cong V$$

Let $\alpha_1, \ldots, \alpha_n$ be simple roots for W acting on V

$\tilde{\alpha}$ = highest root of W

Lemma. $D \cap E = \left\{ e_0 + z \;\middle|\; \begin{array}{l} (z, \alpha_i) \geq 0 \text{ for } i = 1, \ldots, n \\ (z, \tilde{\alpha}) \leq 1 \end{array} \right\}$

Pf. Pick $e_0 + z \in D \cap E$. Then $(e_0+z)(\alpha_s) \geq 0$ for all $\alpha_s \in S_a$.

& $(e_0+z)(\delta) = 1$. Note: $e_0(\alpha_s) = 0$ for $s \in S$, so $z(\alpha_s) \geq 0$ for $s \in S$.

$e_0(\alpha_{s_0}) = 1 \Rightarrow z(\alpha_{s_0}) \geq -1$. When we identify $(V_a/V_a^{\perp})^* \cong V$,

α_s $(s \in S)$ becomes α_i for some $i = 1, \ldots, n$ and α_{s_0} becomes $-\tilde{\alpha}$,

so $\Rightarrow (z, \alpha_i) \geq 0$, $i = 1, \ldots, n$, & $(z, \tilde{\alpha}) \leq 1$.

Conversely, if these hold, then $(e_0+z)(\alpha_s) \geq 0$, for $s \in S$ and

$$1 = (e_0+z)(\delta) = (e_0+z)(\alpha_{s_0}) + \underbrace{z(\delta - \alpha_{s_0})}_{(z, \tilde{\alpha})} \Rightarrow (e_0+z)(\alpha_{s_0}) \geq 0. \quad \square$$

Define $A_0 = \{ v \in V \mid (v, \alpha_i) > 0 \;\; i = 1, \ldots, n, \;\; (v, \tilde{\alpha}) < 1 \}$

$A = \{ v \in V \mid (v, \alpha_i) \geq 0 \;\; i = 1, \ldots, n, \;\; (v, \tilde{\alpha}) \leq 1 \}$

<u>fundamental alcove</u> (open simplex), $A = \bar{A}_0$.

Each W_a-orbit in V intersects A in at most one point.

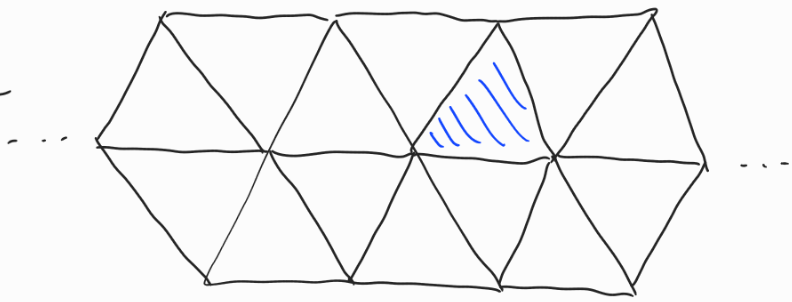
Prop. The W_a-orbit of A is V.

Pf. Pick $\lambda \in A_0$. Pick any $\mu \in V$. Note: $T_{L^\vee}\mu$ is a discrete subset of V, since $W_a / T_L v \cong W$ is finite, $W_a \cdot \mu$ is also discrete. So $\exists\, \nu \in W_a \cdot \mu$ which minimizes distance to λ. If $\nu \in A$, done. Otherwise, $\exists$ hyperplane that bounds A and separates λ, ν. Reflect across hyperplane to get point closer to λ. But this reflection belongs to W_a $\rightarrow\!\leftarrow$ $\square$

<u>Ex</u>. A_1 root system. $V \cong \mathbb{R}$, $\Phi = \{2, -2\}$, so $\alpha_1^\vee = 1$

$A_0 = (0,1)$

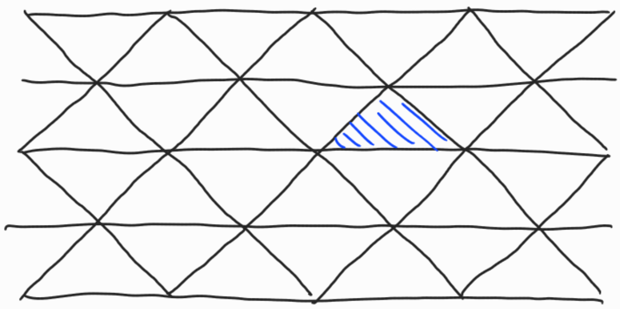
$\tilde{A}_2$



$x_1 - x_2 \geq 0$
$x_2 - x_3 \geq 0$
$x_1 - x_3 \leq 1$
$x_1 + x_2 + x_3 = 0$

$\alpha_1 = e_1 - e_2$
$\alpha_2 = e_2 - e_3$
$\tilde{\alpha} = e_1 - e_3$

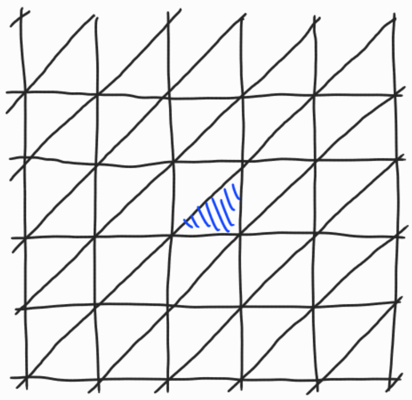
$\tilde{B}_2$



$x - y \geq 0$
$y \geq 0$
$x + y \leq 1$

$\alpha_1 = e_1 - e_2$
$\alpha_2 = e_2$
$\tilde{\alpha} = e_1 + e_2$

$\tilde{C}_2$



$x - y \geq 0$
$y \geq 0$
$x \leq 1/2$

$\alpha_1 = e_1 - e_2$
$\alpha_2 = 2e_2$
$\tilde{\alpha} = 2e_1$

<u>Length function</u> $w \in W_a$. $H = H_{\alpha,k}$ separates A_0 and wA_0 if they are on different sides of H.

$\mathcal{L}(w) = \{ H_{\alpha,k} \mid H_{\alpha,k} \text{ separates } A_0 \text{ \& } wA_0 \}$

$H_{s_0} = H_{\tilde{\alpha},1}$, $H_s = H_{s,0}$ for $s \in S$.

<u>Lemma.</u> For $s \in S_a$, $\mathcal{L}(s) = \{H_s\}$.

<u>Pf.</u> Clear that $H_s \in \mathcal{L}(s)$. Pick $x \in A_0$. Then $0 < (x, \alpha) < 1$ for all $\alpha \in \Phi^+$. Pick $t \in S_a \setminus \{s\}$. Then $(sx, \alpha_t) = (x, s\alpha_t)$ and $s\alpha_t > 0$, so $0 < (sx, \alpha_t) < 1$. $\Rightarrow$ A_0 and sA_0 are on same side of $H_{\alpha,k}$ for any $\alpha \neq \alpha_s$. Since $(xs, \alpha_s) = -(x, \alpha_s)$ and $0 < (x, \alpha_s) < 1$, we see that $H_{\alpha_s,k}$ separates only for $k=0$. $\square$

<u>Lemma.</u> Pick $w \in W_a$, $s \in S_a$.

(1) H_s is in exactly one of $\mathcal{L}(w^{-1})$ or $\mathcal{L}(sw^{-1})$

(2) $s(\mathcal{L}(w^{-1}) \setminus \{H_s\}) = \mathcal{L}(sw^{-1}) \setminus \{H_s\}$.

<u>Pf.</u> (1) If $x \in w^{-1}A_0$, then $(\alpha_s, sx) = -(\alpha_s, x)$, so $w^{-1}A_0$ and $sw^{-1}A_0$ are on different sides of H_s.

(2) Suppose $H \in \mathcal{L}(w^{-1}) \setminus \{H_s\}$. Since $sH_s = H_s$, we know $sH \neq H_s$

We claim $sH \in \mathcal{L}(sw^{-1})$. If not, then $sw^{-1}A_0$ and A_0 are on same side of sH. $\Rightarrow$ $w^{-1}A_0$ and sA_0 same side of H.

We know H separates $w^{-1}A_0$ and A_0. $\Rightarrow$ H separates A_0 and sA_0

By previous lemma $\Rightarrow$ $H = H_s$. $\rightarrow\leftarrow$.

$\Rightarrow s(\mathcal{L}(w^{-1}) \setminus \{H_s\}) \subseteq \mathcal{L}(sw^{-1}) \setminus \{H_s\}$

Reverse inclusion follows by symmetry. $\square$

Prop. Pick reduced expression $w = s_{i_1} \cdots s_{i_r}$ for $w \in W_a$.

(1) The hyperplanes $H_{s_{i_1}}, \; s_{i_1} H_{s_{i_2}}, \; \dots \; s_{i_1} \cdots s_{i_{r-1}} H_{s_{i_r}}$ are distinct

(2) $\mathcal{L}(w) = \{$

Pf. (1) If not, $\exists\, p < q$ s.t. $s_{i_1} \cdots s_{i_{p-1}} H_{s_{i_p}} = s_{i_1} \cdots s_{i_{q-1}} H_{s_{i_q}}$

$\Rightarrow H_{s_{i_p}} = s_{i_p} \cdots s_{i_{q-1}} H_{s_{i_q}}$. From last lecture, we know $\exists$

$x \in L^\vee$ and $u \in W$ s.t. $s_{i_p} \cdots s_{i_{q-1}} = t_x \cdot u$.

$\Rightarrow (s_{i_p} \cdots s_{i_{q-1}}) s_{i_q} (s_{i_p} \cdots s_{i_{q-1}})^{-1} = s_{i_p}$.

$\Rightarrow s_{i_p} \cdots s_{i_{q-1}} s_{i_q} = s_{i_{p+1}} \cdots s_{i_{q-1}} \rightarrow\leftarrow$ reduced expression for w

(2) Induction on $\ell(w)$. $\ell(w) = 0$ clear ✓

Assume $\ell(w) > 0$. By induction

$$\mathcal{L}(s_{i_1} w) = \{ H_{s_{i_2}}, s_{i_2} H_{s_{i_3}}, \dots, s_{i_2} \cdots s_{i_{r-1}} H_{s_{i_r}} \}$$

and by (1), set has size $r-1$. Apply s_{i_1} to all to set

another set of $r-1$ hyperplanes, not containing $H_{s_{i_1}}$ by (1)

$\Rightarrow \mathcal{L}(s_{i_1} w) \not\ni H_{s_{i_1}}$. By previous lemma, $H_{s_{i_1}} \in \mathcal{L}(w)$ and

$s_{i_1} (\mathcal{L}(w) \setminus \{H_{s_{i_1}}\}) = \mathcal{L}(s_{i_1} w)$. $\square$

Cor. For $w \in W_a$, $\ell(w) = |\mathcal{L}(w)|$.