Examples of complex reflection groups

Lemma. $h_1, \dots h_n$ homogeneous polynomials s.t. $h_1(\alpha) = \dots = h_n(\alpha) = 0$ has only solution $\alpha = 0$. Then h_i are algebraically independent.

Pf. (Sketch) Let I = ideal generated by h_i.

Nullstellensatz $\Longrightarrow$ radical of I = ideal gen by variables

$\Longrightarrow$ A/I is finite dimensional

Pick basis for A/I, their preimages in A give a set of generators for A as a module over $\mathbb{C}[h_1, \dots, h_n]$

Set of generators is finite $\Longrightarrow$ $\mathrm{Frac}(A)$ is finite dim over $\mathbb{C}(h_1, \dots, h_n)$

$\Longrightarrow$ same transcendence degree over $\mathbb{C}$ $\Longrightarrow$ $h_1, \dots, h_n$ are algebraically independent.

Rmk. Condition that $h_1, \dots h_n$ only have trivial solution is called "system of parameters", is stronger than alg. ind.

x^2, xy has nontrivial solution $(0,1)$ but are algebraically independent: $J \underset{\text{def}}{=} \begin{pmatrix} 2x & 0 \\ y & x \end{pmatrix} = 2x^2 \neq 0$.

Note: Suppose we have alg. ind. elements $f_1, \dots f_n$ s.t. $(\deg f_1) \cdots (\deg f_n) = |W|$. ($W$ = reflection group). Then, $f_1, \dots, f_n$ generate A^W.

m, p, n be positive integers s.t. $p \mid m$.

$G(m,p,n)$ = group of $n \times n$ matrices which
- have one nonzero entry in each row and column
- nonzero entries are m^{th} roots of unity
- product of nonzero entries is an $(m/p)^{th}$ root of unity.

Size: $|G(m,1,n)| = n!\, m^n$

$$|G(m,p,n)| = \frac{n!\, m^n}{p}$$

<u>Generators</u>: let $\omega = \exp(2\pi i/m)$ primitive mth root of unity.

<u>$p=1$</u>: $G(m,1,n)$ is generated by $\underbrace{\mathfrak{S}_n \text{ (permutation matrices)}}$ + $\begin{pmatrix} \omega & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}$

generated by transpositions $(i, i+1)$ for $i=1,\dots,n-1$

<u>$p=m$</u>: If $n=1$, $G(m,m,1)$ = trivial group

Assume $n>1$.

$G(m,m,n)$ is generated by $\mathfrak{S}_n$ + $\begin{pmatrix} 0 & \omega & & \\ \omega^{-1} & 0 & & 0 \\ & & 1 & \\ & 0 & & \ddots \\ & & & & 1 \end{pmatrix}$

eigenvalues are $-1, 1, \dots 1$

<u>General p</u>: $G(m,p,n)$ is generated by $G(m,m,n)$ + $\begin{pmatrix} \omega^p & & 0 \\ & 1 & \\ 0 & & \ddots \\ & & & 1 \end{pmatrix}$

<u>To see this</u>: $G(m,p,n)$ is gen. by $\mathfrak{S}_n$ and $\begin{pmatrix} \omega^{a_1} & & 0 \\ & \ddots & \\ 0 & & \omega^{a_n} \end{pmatrix}$ where $a_i \in \mathbb{Z}/m$

and $\sum_{i=1}^{n} a_i = 0 \pmod p$

<u>Real reflection groups</u>: • If $m=1$, $G(1,1,n) = \mathfrak{S}_n$ (type A_{n-1})

• $m=2$, $p=1$, $G(2,1,n) = \mathfrak{S}_n \ltimes (\mathbb{Z}/2)^n$ (type B_n)

• $m=2$, $p=2$, $G(2,2,n)$ = (type D_n)

• $m=p$, $n=2$, $G(m,m,2)$ = dihedral group of order $2m$. (type $I_2(m)$)

<u>Generators</u> $\mathfrak{S}_n = G(1,1,n)$

<u>Recall</u>: $e_p(X) = \sum_{1 \le i_1 < \cdots < i_p \le n} X_{i_1} \cdots X_{i_p}$ (elementary symmetric polynomial)

<u>Note</u>: $\sum_{i=0}^{n} (-1)^p e_p(x) t^{n-p} = (t-x_1)\cdots(t-x_n)$

If $e_p(\alpha)=0$ for $p=1,\dots,n$, then $(t-\alpha_1)\cdots(t-\alpha_n) = t^n \Rightarrow \alpha = 0$

$\Rightarrow e_1,\dots,e_n$ alg. ind., $1 \cdot 2 \cdots n = |\mathfrak{S}_n|$, so they generate $A^{\mathfrak{S}_n}$.

$\underline{G(m,1,n)}$ Take $e_p(x_1^m, \dots, x_n^m)$ for $p=1,\dots,n$

degrees are $m, 2m, \dots nm$, product is $m^n n! = |G(m,1,n)|$

If $e_p(\alpha^m)=0$ for $p=1,\dots n$, then $\alpha_1^m = \dots = \alpha_n^n = 0 \Rightarrow \alpha = 0$

$e_p(x^m)$ generates $A^{G(m,1,n)}$.

$\underline{G(m,p,n)}$ Take $\left.\begin{array}{l} e_p(x_1^m, \dots, x_n^m) \text{ for } p=1,\dots,n-1 \\ (x_1 \cdots x_n)^{m/p} \end{array}\right\}$ generate $A^{G(m,p,n)}$.

degrees are $m, 2m, \dots, (n-1)m, \frac{nm}{p}$, product is $\frac{m^n n!}{p} = |G(m,p,n)|$

Shepherd-Todd classification of irreducible complex reflection groups: $G(m,p,n)$ + 34 exceptional cases

$\underline{\text{Ex}}$. (group #24) Klein quartic

$$x^3y + y^3z + z^3x = 0$$

(projective) solution set is a genus 3 Riemann surface

Its automorphism group has size 168

is

$$SL_2(\mathbb{F}_7) = \left\{ \begin{array}{l} 2\times 2 \text{ matrices w/ entries} \\ \text{in } \mathbb{Z}/7 \text{ whose det is } 1 \end{array} \right\}$$

$$PSL_2(\mathbb{F}_7) = SL_2(\mathbb{F}_7)/\{\pm I\}$$

group #24 $\cong PSL_2(\mathbb{F}_7) \times \mathbb{Z}/2$.

degrees of invariants are 4, 6, 14

$\underline{\text{Example \#25}}$. Hesse pencil: ($\lambda, \mu$ projective parameters)

$$\lambda(x_0^3 + x_1^3 + x_2^3) + \mu\, x_0 x_1 x_2 = 0$$

Indices are elements of $\mathbb{Z}/3$. let $\omega = \exp(2\pi i/3)$

Define $\sigma(x_i) = x_{i+1}$, $\tau(x_i) = \omega^i x_i$, σ, τ generate a nonabelian group H of order 27 sits in $GL_3\mathbb{C}$

group #25 N normalizes H & $N/H \cong SL_2(\mathbb{F}_3)$

12 reflection planes: there are 4 values $[\lambda : \mu]$ s.t. curve in pencil is singular (= union of 3 projective lines) Union of these lines gives reflection planes