Coinvariant ring. $W \subset GL_n\mathbb{C}$ complex reflection group
$A = \mathbb{C}[x_1, \dots x_n]$, I = ideal generated by W-invariants $f_1, \dots, f_n$

Coinvariant ring is A/I

Pick a homogeneous basis $\{\overline{v_i}\}$ for A/I. Choose homogeneous reps v_i for $\overline{v_i} + I$.

Lemma. $\{v_i\}$ give basis for A as an A^W-module.

Pf. Pick $f \in A$ homogeneous. Prove that $f \in A^W$-module generated by v_i by induction on $\deg f$. $\deg f = 0$ ✓

Assume $\deg f > 0$. Let $\overline{f}$ be image of f in A/I.

$\exists$ scalars c_i s.t. $\overline{f} = \sum_i c_i \overline{v_i}$. $\Rightarrow f - \sum_i c_i v_i \in I$.

$\exists$ homogeneous h_j s.t. $f - \sum_i c_i v_i = \sum_{j=1}^n h_j f_j$.

$\Rightarrow \deg f = \deg h_j + \deg f_j \ \forall j \Rightarrow \deg h_j < \deg f$ for all j.

By induction, h_j is in A^W-module generated by $\{v_i\}$

$\Rightarrow f = \sum_i c_i v_i + \sum_{j=1}^n h_j f_j$ is in A^W-module generated by $\{v_i\}$.

We Claim: If $u_1, \dots u_m \in A$ homogeneous s.t. their images in A/I are linearly independent, then $u_1, \dots u_m$ are linearly independent over A^W.

Induction on m: $m=1$ ✓

Assume $m > 1$, $u_1, \dots u_m$ are dependent over A^W, but images lin. ind. in A/I.

$\Rightarrow \exists h_1, \dots h_m \in A^W$ (homog.) s.t. $h_1 u_1 + \dots + h_m u_m = 0$

$u_1 \notin I$ (by lemma from last lecture), $h_1 \overset{A^W}{\in}$ ideal generated by $h_2, \dots, h_m \Rightarrow h_1 = \alpha_2 h_2 + \dots + \alpha_m h_m$, $\alpha_i \in A^W$.

$\Rightarrow (\alpha_2 h_2 + \dots + \alpha_m h_m) u_1 + h_2 u_2 + \dots + h_m u_m = 0$

$\Rightarrow h_2(u_2 + \alpha_2 u_1) + \dots + h_m(u_m + \alpha_m u_1) = 0$

$$u_i + \alpha_i u_1 \equiv u_i \mod I$$

$\Rightarrow u_2 + \alpha_2 u_1, \dots, u_m + \alpha_m u_1$ have linearly independent images in A/I

By induction on m, they are independent over $A^W \Rightarrow h_2 = \dots = h_m = 0$

$\Rightarrow h_1 = 0$

$\Rightarrow u_1, \dots, u_m$ linearly independent over A^W. $\square$

Cor ① A/I is a vector space of dimension $|W|$.

② A is free as A^W-module of rank $|W|$.

③ A/I is regular representation of W.

Pf. Claim: $\mathrm{Frac}(A) = A \otimes_{A^W} \mathrm{Frac}(A^W)$

$\mathrm{Frac}(A)$ is finite extension over $\mathrm{Frac}(A^W)$ of degree $|W|$.

If $f \in A$, then $\frac{1}{f}$ satisfies a monic polynomial w/ coeff in $\mathrm{Frac}(A^W)$:

$$\left(\frac{1}{f}\right)^n + a_{n-1}\left(\frac{1}{f}\right)^{n-1} + \dots + a_0 = 0, \quad a_i \in \mathrm{Frac}(A^W).$$

Multiply by f^{n-1}:

$\in A \otimes \mathrm{Frac}(A^W)$

$$\frac{1}{f} = -(a_{n-1} + \dots + a_0 f^{n-1}) \Rightarrow \frac{1}{f} \in A \otimes \mathrm{Frac}(A^W)$$

$\Rightarrow A \otimes \mathrm{Frac}(A^W) = \mathrm{Frac}(A)$.

$$\dim_{\mathbb{C}}(A/I) = \mathrm{rk}_{A^W} A = \dim_{\mathrm{Frac}(A^W)} \mathrm{Frac}(A) = |W| \Rightarrow ① + ②.$$

③ Pick basis $\bar{v}_1, \dots, \bar{v}_N$ for A/I, lift to $v_1, \dots, v_N$ basis for A as A^W-module. $\{v_i\}$ also basis for $\mathrm{Frac}(A)$ as $\mathrm{Frac}(A)^W$-vector space.

NORMAL BASIS THM: $\mathrm{Frac}(A)$ is regular rep. for W as $\mathrm{Frac}(A)^W$-vector space.

$\Rightarrow$ Every $w \in W$, $w \neq 1$ has trace 0 (wrt v_i)

$\Rightarrow$ Reduce mod I, $w \neq 1$ has trace 0 on A/I.

$\Rightarrow$ A/I = regular rep. for W. $\square$

<u>Cor.</u> Let $d_i = \deg(f_i)$. Then

$$H_{A/I}(t) = \frac{H_A(t)}{H_{A^W}(t)} = \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t} = \prod_{i=1}^{n} \left(1+t+\cdots+t^{d_i-1}\right)$$

<u>Pf.</u> First result $\Rightarrow$ $H_A(t) = H_{A^W}(t) H_{A/I}(t)$

$H_A(t) = \frac{1}{(1-t)^n}$, Chevalley $\Rightarrow$ $H_{A^W}(t) = \prod_{i=1}^{n} \frac{1}{1-t^{d_i}}$ $\square$