Chevalley-Shepherd-Todd Theorem

Def. $g \in GL_n\mathbb{C}$ is a reflection if $\operatorname{rank}(g-I)=1$ & g has finite order.

$W \subseteq GL_n\mathbb{C}$ is a finite group generated by reflections

complex reflection group

$A = \mathbb{C}[x_1, \dots, x_n]$, $I =$ ideal gen. by A^W.

Lemma Let $f_1, \dots f_k \in A^W$ (homog) s.t. f_1 is not in ideal of A^W gen. by $f_2, \dots f_k$. Suppose $\exists$ homog. $h_i \in A$ s.t.

$(*)$ $f_1 h_1 + \dots + f_k h_k = 0$ Then $h_1 \in I$.

Pf. Induction on $\deg h_1$. Base case: $\deg h_1 = 0$, h_1 constant.

Apply $\#$ to $(*)$: $f_1 h_1 = -(h_2^\# f_2 + \dots + h_k^\# f_k)$.

If $h_1 \neq 0$, then $f_1 \in A^W$-ideal gen. by $f_2, \dots f_k \rightarrow\leftarrow \Rightarrow h_1 = 0 \in I$.

Now suppose $\deg h_1 > 0$. Let $s \in W$ be reflection, let $l_s =$ linear function cutting out the fixed hyperplane H_s of s. For each i, $sh_i - h_i$ is 0 on $H_s \Rightarrow sh_i - h_i = h_i' l_s$. If we apply s to $(*)$, get

$f_1(sh_1) + \dots + f_k(sh_k) = 0$. Subtract from $(*)$ to get

$l_s(f_1 h_1' + \dots + f_k h_k') = 0 \Rightarrow f_1 h_1' + \dots + f_k h'_k = 0$

$\deg h_i' = \deg h_i - 1 \Rightarrow$ By induction, $h_1' \in I$.

$\Rightarrow sh_1 - h_1 \in I \Rightarrow sh_1 \equiv h_1 \pmod I$. True for every reflection, $\Rightarrow gh_1 \equiv h_1 \pmod I$ for any $g \in W \Rightarrow h_1^\# = h_1 \pmod I$

$\Rightarrow h_1 \in I$. $\square$

Lemma If f homog poly in $x_1, \dots, x_n$, then

$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = (\deg f) f.$$

Pf. $f = x_1^{p_1} \cdots x_n^{p_n}$, then LHS is $\sum_{i=1}^{n} p_i f = (\deg f) f$. $\square$

Thm (Chevalley) A^W is generated by n alg. independent homog. polynomials.

Pf. Let $f_1, \dots f_k$ be min. homog. generating set for I as A ideal.
From before, $f_1 \dots f_k$ generate A^W as an algebra.

Goal: Show $f_1, \dots f_k$ alg. ind.

Let $y_1 \dots y_k$ be new variables, $\deg y_i = \deg f_i$

$$\psi : \mathbb{C}[y_1, \dots, y_k] \longrightarrow \mathbb{C}[x_1, \dots x_n]$$
$$\psi(h(y_1, \dots, y_k)) = h(f_1, \dots f_k)$$

ψ is injective $\Leftrightarrow$ $f_1, \dots f_k$ alg. ind.

Suppose not, $\ker \psi$ is homogeneous nonzero ideal. Pick $h \in \ker \psi$ homog., minimal possible degree. Apply $\frac{\partial}{\partial x_i}$ to $h(f_1, \dots f_k) = 0$:

$$\sum_{j=1}^{k} h_j \frac{\partial f_j}{\partial x_i} = 0 \qquad \text{where } h_j = \frac{\partial h}{\partial y_j}(f_1, \dots, f_k)$$

Claim. At least one $h_j \neq 0$.

Pf. $\deg h > 0$, some $\frac{\partial h}{\partial y_j}$ is nonzero, so $\frac{\partial h}{\partial y_j} \notin \ker \psi$!

Renumber $h_1 \dots h_k$ so that $h_1, \dots h_m$ are a minimal generating set for the ideal in A^W gen by $h_1, \dots h_k$. $(m > 0)$

For $m+1 \leq j \leq k$, we can write $h_j = \sum_{a=1}^{m} \alpha_{j,a} h_a$

$\alpha_{j,a} \in A^W$, homogeneous of degree $\deg(h_j) - \deg(h_a) = \deg(f_a) - \deg(f_j)$ (if nonzero)

$$\sum_{i=1}^{m} h_j' \, p_{ij} = 0 \quad \text{where} \quad p_{ij} = \frac{\partial f_j}{\partial x_i} + \sum_{b=m+1}^{k} \alpha_{bj} \frac{\partial f_b}{\partial x_i}$$

$\xRightarrow{\text{Lemma}}$ $p_{i1} \in I$ (since h_1 not in A^W-ideal gen. by $h_2, \ldots h_m$)

$$I \ni \sum_{i=1}^{n} x_i p_{i1} = \sum_{i=1}^{n} x_i \frac{\partial f_1}{\partial x_i} + \sum_{b=m+1}^{k} \alpha_{b1} \sum_{i=1}^{n} x_i \frac{\partial f_b}{\partial x_i}$$

$$\overset{\text{Euler}}{=} \underbrace{(\deg f_1) f_1 + \sum_{b=m+1}^{k} \alpha_{b1} (\deg f_b) f_b}_{\text{homog. of deg } = \deg f_1}$$

$\sum_{j=1}^{k} \beta_j f_j$ If $\beta_1 \neq 0$, then $\deg \beta_1 > 0$

can rewrite so $\beta_1 = 0$

$\Rightarrow$ f_1 can be expressed as linear comb. of $f_2, \ldots f_k$ $\rightarrow\leftarrow$ $\Rightarrow$ $\ker \psi = 0 \Rightarrow f_1, \ldots f_k$ are alg. ind.

$\Rightarrow A^W \cong \mathbb{C}[y_1, \ldots y_k]$

$k = \operatorname{trdeg} \operatorname{Frac}(A^W) = \operatorname{trdeg} \mathbb{C}(x_1, \ldots, x_n) = n$ $\square$

Rmk. $f_1, \ldots f_n$ are not unique, but their degrees are:

$$H_{A^W}(t) = \prod_{i=1}^{n} \frac{1}{1 - t^{\deg f_i}}$$

Thm. $G \subset GL_n\mathbb{C}$ finite group, assume that A^G is gen. by alg. ind. homog polys $f_1, \ldots f_n$. $d_i := \deg f_i$. Let T = set of reflections in G. Then, $|G| = d_1 \cdots d_n$ & $d_1 + \cdots + d_n = |T| + n$.

Pf.
$$H_{A^G}(t) = \prod_{i=1}^{n} \frac{1}{1 - t^{d_i}} \underset{\text{Molien}}{=} \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1 - gt)}$$

$\det(1-gt) = \prod_{i=1}^{n}(1-\lambda_i t)$ where $\lambda_1, \dots \lambda_n$ are eigenvalues of g.

If $g=1$, then get $(1-t)^n$

If $g \in T$, then get $(1-t)^{n-1}(1-\omega t)$ where ω root of unity $\neq 1$.

If $g \notin T$, $g \neq 1$, then get $\det(1-gt)$ not divisible by $(1-t)^{n-1}$

Multiply by $(1-t)^n$

$$\prod_{i=1}^{n} \frac{1}{1+t+t^2+\dots+t^{d_i-1}} = \frac{1}{|G|}\left(1 + \sum_{g\in T}\frac{1-t}{1-\omega_g t} + (1-t)^2 F(t)\right)$$

rational function w/ no pole at $t=1$

Plug in $t=1$:

$$\prod_{i=1}^{n}\frac{1}{d_i} = \frac{1}{|G|}(1+0+0) \Rightarrow d_1 \cdots d_n = |G|$$

$$\left(\prod f_i(t)\right)' = \prod f_i(t) \sum \frac{f_i'(t)}{f_i(t)}$$

Take derivative:

$$\left(\prod_{i=1}^{n}\frac{1}{1+t+t^2+\dots+t^{d_i-1}}\right)' = \left(\prod_{i=1}^{n}\frac{1}{1+t+\dots+t^{d_i-1}}\right)\sum_{i=1}^{n}\frac{-(1+2t+\dots+(d_i-1)t^{d_i-2})}{(1+t+\dots+t^{d_i-1})}.$$

$$\left(\frac{1}{|G|}\left(1+\sum_{g\in T}\frac{1-t}{1-\omega_g t} + (1-t)^2F(t)\right)\right)' = \frac{1}{|G|}\left(\sum_{g\in T}\frac{\omega_g - 1}{(1-\omega_g t)^2} + H(t)\right)$$

where $H(t) = \left((1-t)^2 F(t)\right)'$

and $H(1)=0$

Plug in $t=1$:

$$\cancel{-}\cancel{\prod_{i=1}^{n}\frac{1}{d_i}}\sum_{i=1}^{n}\frac{d_i(d_i-1)/2}{\cancel{d_i}} = \cancel{-}\cancel{\frac{1}{|G|}}\sum_{g\in T}\frac{1}{1-\omega_g} = \sum_{\substack{g\in T\\ \text{minimal}}} \boxed{\sum_{i=1}^{\operatorname{ord}(g)-1}\frac{1}{1-\omega_g^i}} = \sum_{g\in T}\frac{1}{2}$$

By next lemma, $= \frac{\operatorname{ord}_g - 1}{2}$

$$\| $$

$$\frac{1}{2}\sum_{i=1}^{n}(d_i-1)$$

If $g\in T$ has order r, then $g, g^2, \dots g^{r-1} \in T$ and $\omega_{g^i} = \omega_g^i$. Call $g \in T$ minimal if g not a power of another reflection

$$\Rightarrow \frac{1}{2}\sum_{i=1}^{n}(d_i-1) = \frac{1}{2}|T| \Rightarrow d_1+\cdots+d_n = |T|+n \qquad \square$$

Lemma. ω = primitive rth root of unity. Then

$$\sum_{i=1}^{r-1}\frac{1}{1-\omega^i} = \frac{r-1}{2}.$$

Pf. $f(x) = \prod_{i=1}^{r-1}(x-\omega^i) = \frac{x^r-1}{x-1} = \underline{1+x+\cdots+x^{r-1}}$

$$f'(x) = \prod_{i=1}^{r-1}(x-\omega^i)\sum_{i=1}^{r-1}\frac{1}{x-\omega^i}$$

$$\frac{f'(x)}{f(x)} = \sum_{i=1}^{r-1}\frac{1}{x-\omega^i}$$

Plug in $x=1$: $f'(x) = 1+2x+\cdots+(r-1)x^{r-2}$

$f'(1) = 1+2+\cdots+r-1 = \frac{r(r-1)}{2}$

$f(1) = r$

$$\Rightarrow \frac{r-1}{2} = \sum_{i=1}^{r-1}\frac{1}{1-\omega^i}. \qquad \square$$

Thm (Shephard-Todd). Let $G \subset GL_n\mathbb{C}$ finite, suppose A^G is gen by alg. independent elements. Then G is generated by reflections.

Pf. Let $H \subset G$ be subgroup gen. by $\underbrace{\text{all reflections in } G}_{T}$.

Chevalley $\Rightarrow$ A^H gen by alg. ind. elements of degrees $e_1 \le e_2 \le \cdots \le e_n$.

Let $d_1 \le \cdots \le d_n$ be degrees of generators for A^G.

Since $A^G \subseteq A^H$ we must have $e_i \le d_i$ for all i.

$$d_1+\cdots+d_n = |T|+n = e_1+\cdots+e_n \Rightarrow e_i = d_i \text{ for all } i$$

$$|H| = e_1\cdots e_n = d_1\cdots d_n = |G| \Rightarrow H = G. \qquad \square$$