Hyperbolic Coxeter Groups

B_W is nondegenerate, $n = |S|$

$\rightsquigarrow$ dual basis ω_s to α's defined $B_W(\omega_s, \alpha_t) = \delta_{st}$.

$$C = \{ v \in V \mid B_W(v, \alpha_t) > 0 \text{ for all } t \in S \}$$
$$= \{ \sum_s c_s \omega_s \mid c_s > 0 \text{ for all } s \in S \}.$$

Recall: every real bilinear form is equivalent to $x_1^2 + \dots + x_p^2 - (x_{p+1}^2 + \dots + x_{p+q}^2)$ and signature is (p, q)

Def. (W, S) is hyperbolic if B_W has signature $(n-1, 1)$ & $B_W(v, v) < 0$ $\forall$ $v \in C$.

Lemma E = real n-dim vector space, B symmetric bilinear form on E of signature $(n-1, 1)$. Pick nonzero $v \in E$, $v^\perp = \{ w \in E \mid B(v, w) = 0 \}$.

Then B restricted to $v^\perp$ is positive semidefinite $\Longleftrightarrow$ $B(v, v) \leq 0$.

Pf. Suppose $B(v, v) \neq 0$. Then $v \notin v^\perp$, so $E = \langle v \rangle \oplus v^\perp$ (orthogonal)

B is direct sum of its restrictions to $\langle v \rangle$ and $v^\perp$.

$B(v, v) < 0 \Longleftrightarrow$ signature of B on $v^\perp$ is $(n-1, 0)$ pos. def. ✓

$B(v, v) > 0 \Longleftrightarrow$ signature of B on $v^\perp$ is $(n-2, 1)$ not pos. semidef ✓

Now suppose $B(v, v) = 0$, so $v \in v^\perp$. Since B has signature $(n-1, 1)$, $\exists$ hyperplane H on which it is pos. def, $H \neq v^\perp$

$\Rightarrow$ B is pos. def. on $H \cap v^\perp$ $\Rightarrow$ signature on $v^\perp$ is $(n-2, 0)$

(pos. semidef.) ✓ $\square$

Thm. Γ = Coxeter graph of (W, S). (W, S) hyperbolic $\Longleftrightarrow$

(a) B_W is nondegenerate, but not pos. definite.

(b) $\forall s \in S$, $\Gamma \setminus s$ is positive semidefinite.

$s \in S$, L_s = span of $\{ \alpha_t \mid t \neq s \}$

Pf. Suppose (W,S) hyperbolic. (a) by definition

(b) Bilinear form on $P\setminus s$ is $B_W|_{L_s}$, $L_s = \omega_s^\perp$

$B_W|_{L_s}$ pos. semidef $\iff B_W(\omega_s,\omega_s) \leq 0$

Note: $x \mapsto B_W(x,x)$ is continuous, ω_s is in closure of C,

so $B_W(\omega_s,\omega_s) \leq 0$.

Conversely, suppose (a) + (b) hold.

Define $N = \{v \in V \mid B_W(v,v) < 0\}$

By (a), $N \neq \emptyset$. By (b) $N \cap L_s = \emptyset$

$\Rightarrow N \subset V \setminus \bigcup_{s\in S} L_s$ $\leftarrow$ disconnected, one conn. comp U_T for each subset $T \subset S$: $U_T = \left\{\sum c_s \alpha_s \;\middle|\; \begin{array}{l} c_s > 0 \text{ if } s \in T \\ c_s < 0 \text{ if } s \notin T \end{array}\right\}$

Suppose signature of B_W is (p,q) w/ $q \geq 2$.

$\Rightarrow$ 2-dim subspace Z s.t. $B_W|_Z$ is negative definite

i.e., $Z \setminus 0 \subset N$, $Z\setminus 0 \cong \mathbb{R}^2 \setminus 0$, connected

$\Rightarrow \exists T$ s.t. $Z\setminus 0 \subset U_T$, but $Z \setminus 0$ closed under negation, but U_T is not $\rightarrow\leftarrow$

$\Rightarrow$ Signature must be $(n-1,1)$.

Remains to show that $C \subset N$.

Note. in some basis, $B_W(x,x) = x_1^2 + \cdots + x_{n-1}^2 - x_n^2$

so $N = \{x \mid x_1^2 + \cdots + x_{n-1}^2 < x_n^2\}$

2 connected components corresponding to sign of x_n

Furthermore each component is closed under addition.

By (b), $\forall s \in S$, B_W is pos. semidef on $L_s = \omega_s^\perp \Rightarrow B_W(\omega_s,\omega_s) \leq 0$

so ω_s belongs to $\overline{N}$

Claim: all ω_s belong to closure of single connected component.

Suppose $B_W(v,v) \le 0$. Write $v = \sum_s c_s \alpha_s$, define $v_+ = \sum_{\substack{s \\ c_s > 0}} c_s \alpha_s$

$v_- = \sum_{\substack{s \\ c_s < 0}} c_s \alpha_s$

<u>Note</u>: $B_W(v_+,v_+) + B_W(v_-,v_-) \le B_W(v,v) = \sum_{s \in S} c_s^2 + \sum_{s \neq t} c_s c_t B_W(\alpha_s,\alpha_t)$

remove terms where $c_s c_t < 0$

$\Rightarrow$ either $B_W(v_+,v_+) \le 0$
or $B_W(v_-,v_-) \le 0$.

$\Rightarrow$ All vectors in N when expanded in α basis, all coeff have same sign $\rightsquigarrow$ describes two components.

Write $\omega_s = \sum_t w_{st} \alpha_t$. Claim: $w_{st} \le 0$ $\forall s,t$.

<u>Note</u>: all w_{st} have same sign.

$(w_{s,t}) = (B_W(\alpha_s,\alpha_t))^{-1}$, so $w_{ss} = \dfrac{\det(B_{P,s}) \geq 0}{\det(B_P) < 0}$

If $B_{P,s}$ pos. def, then $w_{ss} < 0$, then done

If $B_{P,s}$ has det 0, consider

$$1 = B_W(\omega_s, \alpha_s) = \sum_{t \neq s} w_{st} \underbrace{B_W(\alpha_s,\alpha_t)}_{\le 0} \Rightarrow w_{s,t} \le 0$$

$\Rightarrow$ All ω_s belong to closure of single conn. comp. of N.

$\Rightarrow$ $\overline{C}$ belongs closure of single conn. comp. of N

$\Rightarrow$ C belongs to interior of $\overline{N}$ $\Rightarrow C \subset N$ $\square$

<u>Cor</u>. Every connected Coxeter graph w/ 3 vertices is either pos. semidef. or hyperbolic.

Pf. Case 1. $\Gamma =$ 

$$B_W = \begin{pmatrix} 1 & -a & 0 \\ -a & 1 & -b \\ 0 & -b & 1 \end{pmatrix} \qquad a, b \geq 0$$

$\longrightarrow$ eigenvalues are $1, 1 \pm \sqrt{a^2+b^2}$ $\rightsquigarrow$ pos semidef or signature $(2,1)$.

Case 2. $\Gamma =$ 

$$B_W = \begin{pmatrix} 1 & -a & -c \\ -a & 1 & -b \\ -c & -b & 1 \end{pmatrix} \qquad a, b, c \geq \tfrac{1}{2}$$

$$\det B_W = 1 - (a^2+b^2+c^2+2abc) \leq 0$$

w/ equality $\iff a = b = c = \frac{1}{2}$

$\hookrightarrow \Gamma = \tilde{A}_2$ pos semidef

Else, $\det < 0$, trace $> 0 \implies$ signature is $(2,1)$. $\square$