Construction of finite root systems

A **lattice** is a discrete subgroup of $\mathbb{R}^n$ which is isomorphic to $\mathbb{Z}^n$

W is **crystallographic** or is a **Weyl group** if $\exists$ lattice $L \subset V$ (geom. rep) s.t. W preserves L.

Prop. If W preserves a lattice, then $m(s,t) \in \{2,3,4,6\}$ for all $s \neq t$.

Pf. Suppose W preserves lattice L. Pick a basis for V consisting of elements of L. In this basis, every matrix of $w \in W$ has integer entries. In particular, its trace (does not depend on basis) is an integer.

If $|S| \leq 1$, result is clear. If $|S| \geq 2$, pick $s \neq t \in S$.

Restrict to span U of $\{\alpha_s, \alpha_t\}$, from earlier calculation, st is rotation by $\frac{2\pi}{m(s,t)}$ (and we can find complement of U where st act as identity. $\Rightarrow$ $\mathrm{trace}(st) = (\dim V - 2) + 2\cos\left(\frac{2\pi}{m(s,t)}\right) \in \mathbb{Z}$

$\Rightarrow \cos\left(\frac{2\pi}{m(s,t)}\right) \in \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$. Since $m(s,t) < \infty$, it must be one of $\{2,3,4,6\}$. $\square$

This rules out H_3, H_4, $I_2(m)$ for $m=5$, $m \geq 7$

$e_1, \ldots, e_n$ standard basis for $\mathbb{R}^n$,

Given roots α, β we say $\alpha \geq \beta$ if $\alpha - \beta$ is positive root,

In all cases, there is unique maximal root (highest root) $\tilde{\alpha}$.

EXAMPLES

- A_n, $V = \{x \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$

  $\Phi = \{e_i - e_j \mid i \neq j\}$

  $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, n$, $\Phi^+ = \{e_i - e_j \mid i < j\}$

$W \cong \mathfrak{S}_{n+1}$ (symmetric group) $|W| = (n+1)!$, $\tilde{\alpha} = e_1 - e_{n+1}$

- B_n, $V = \mathbb{R}^n$. $\Phi = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm e_i\}$

  $\alpha_i = e_i - e_{i+1}$ for $i = 1, \dots, n-1$, $\alpha_n = e_n$.

  $\Phi^+ = \{e_i \pm e_j \mid i < j\} \cup \{e_i\}$

  $W \cong \mathfrak{S}_n \ltimes (\mathbb{Z}/2)^n$ (hyperoctahedral group)

  $=$ group of $n \times n$ signed permutation matrices

  $|W| = n!\, 2^n$, $\tilde{\alpha} = e_1 + e_2$.

- C_n, $V = \mathbb{R}^n$, $\Phi = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_i\}$

  $\alpha_i = e_i - e_{i+1}$ for $i = 1, \dots, n-1$, $\alpha_n = 2e_n$

  $\Phi^+ = \{e_i \pm e_j \mid i < j\} \cup \{2e_i\}$, $\tilde{\alpha} = 2e_1$.

- D_n $V = \mathbb{R}^n$, $\Phi = \{\pm e_i \pm e_j \mid i \neq j\}$

  $\alpha_i = e_i - e_{i+1}$ for $i = 1, \dots, n-1$, $\alpha_n = e_{n-1} + e_n$.

  $\Phi^+ = \{e_i \pm e_j \mid i < j\}$ $\quad s_{\alpha_n}(x) = (x_1, \dots, x_{n-2}, -x_n, -x_{n-1})$

  $(\mathbb{Z}/2)^{n-1} \subset (\mathbb{Z}/2)^n$ subgroup of $(r_1, \dots, r_n)$ s.t. $r_1 + \dots + r_n \equiv 0 \bmod 2$.

  $W \cong \mathfrak{S}_n \ltimes (\mathbb{Z}/2)^{n-1}$ (demi hyperoctahedral group)

  $|W| = 2^{n-1} n!$ (= signed permutation matrices where -1 appears even # of times)

- E_8, $V = \mathbb{R}^8$. $\Phi = \left\{ x \in \mathbb{R}^8 \;\middle|\; \begin{array}{l} \text{① } x \in \mathbb{Z}^8 \text{ or } 2x \in \mathbb{Z}^8 \text{ \& } 2x \text{ has all odd entries} \\ \text{② } x_1 + \dots + x_8 \text{ is even} \\ \text{③ } x_1^2 + \dots + x_8^2 = 2 \end{array} \right\}$

  $|\Phi| = 240$, $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$

  $\alpha_2 = e_1 + e_2$, $\alpha_3 = e_2 - e_1$, $\alpha_4 = e_3 - e_2$, $\alpha_5 = e_4 - e_3$

  $\alpha_6 = e_5 - e_4$, $\alpha_7 = e_6 - e_5$, $\alpha_8 = e_7 - e_6$

$|W| = 2^{14}\cdot 3^5 \cdot 5^2 \cdot 7 \quad \tilde{\alpha} = e_7 + e_8$

- E_7, $V = \text{span}\{\alpha_1, \dots \alpha_7\}$

  $|W(E_7)| = 2903040$, $\tilde{\alpha} = e_8 - e_7$

- E_6, $V = \text{span}\{\alpha_1, \dots \alpha_6\}$

  $|W(E_6)| = 51840$, $\tilde{\alpha} = \frac{1}{2}(e_1 + e_2 + e_3 + e_4 + e_5 - e_6 - e_7 + e_8)$

Rmk. $W(E_n)$ has close connections to finite matrix groups.

$W(E_7) \cong \mathbb{Z}/2 \times Sp_6(\mathbb{F}_2)$ ← symplectic group preserving symplectic form on $\mathbb{F}_2^6$

- F_4, $V = \mathbb{R}^4$, $\Phi = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{array}{l} \text{(1) } x \in \mathbb{Z}^4 \text{ or } 2x \in \mathbb{Z}^4 \text{ and all entries of } 2x \text{ are odd} \\ \text{(2) } x_1^2 + x_2^2 + x_3^2 + x_4^2 \in \{1, 2\} \end{array} \right\}$

  $|\Phi| = 48$, $\alpha_1 = e_2 - e_3$, $\alpha_2 = e_3 - e_4$, $\alpha_3 = e_4$, $\alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$

  $|W| = 1152$, $\tilde{\alpha} = e_1 + e_2$.

- $I_2(6) = G_2$, $V = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$

  $\Phi = \{x \in V \cap \mathbb{Z}^3 \mid x_1^2 + x_2^2 + x_3^2 \in \{2, 6\}\}$

  $|\Phi| = 12$, $\alpha_1 = e_1 - e_2$, $\alpha_2 = -2e_1 + e_2 + e_3$.

  $W \cong$ dihedral group of order 12, $\tilde{\alpha} = -e_1 - e_2 + 2e_3$

- $W(I_2(m))$ = dihedral group of order $2m$
- $W(H_3)$ = symmetry group of regular icosahedron

  3-dim polytope, built out of 20 triangles, meet 5 at a time at 12 vertices, $W(H_3) \cong \mathbb{Z}/2 \times A_5$ ← alternating group
- $W(H_4)$ = symmetry group of 600-cell: 4-dim polytope whose boundary consists of 600 tetrahedra meeting 5 at a time along common edges.

  $|W(H_4)| = 14400$.