Catalan numbers $C_n = \#$ ways to arrange $n$ pairs of parentheses so that they are balanced.

Ex. $n=3$: ()()(), (())(), ((())), (()()), ()(())

$$C(x) = \sum_{n\geq 0} C_n x^n \qquad (C_0 = 1)$$

Lemma. If $n>0$, then $C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}$

Pf. The first symbol must be left parenthesis. Consider the right parenthesis matching it.

$$( \underbrace{\_\_\_}_{\substack{\text{Some set of parentheses} \\ \text{suppose } i \text{ pairs} \\ \# = C_i}} ) \underbrace{\_\_\_}_{\substack{n-1-i \text{ pairs} \\ \text{of parentheses} \\ \# = C_{n-1-i}}} \quad \text{(both balanced)}$$

$$\Rightarrow C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}. \qquad \square$$

Note $\sum_{i=0}^{n-1} C_i C_{n-1-i}$ is coeff of $x^{n-1}$ in $C(x)^2$

$$C(x) = 1 + \sum_{n\geq 1} C_n x^n = 1 + x \sum_{n\geq 1} \left( \sum_{i=0}^{n-1} C_i C_{n-1-i} \right) x^{n-1} \qquad m = n-1$$

$$= 1 + x \sum_{m\geq 0} \left( \sum_{i=0}^{m} C_i C_{m-i} \right) x^m = 1 + x\, C(x)^2.$$

$$C(x) = 1 + x\, C(x)^2$$

$$x\, C(x)^2 - C(x) + 1 = 0$$

In particular, $C(x)$ is a solution to quadratic eqn

$$x t^2 - t + 1 = 0$$

$\Rightarrow$ $C(x)$ is given by quadratic formula, and is

$$\frac{1 \pm \sqrt{1-4x}}{2x}$$ for some choice of sign.

$x$ not invertible $\Rightarrow$ numerator cannot have constant term

$\Rightarrow$ must be negative sign

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}.$$ Let's use binomial Thm.

$$1-(1-4x)^{1/2} = -\sum_{n\geq 1} \binom{1/2}{n} (-4x)^n$$

$$-(-1)^n 4^n \binom{1/2}{n} = -(-1)^n 4^n \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(\frac{-3}{2}\right)\cdots\frac{-(2n-3)}{2}}{n!}$$

$$= 2^n \frac{1\cdot 1\cdot 3\cdots(2n-3)}{n!} = 2^n \frac{(2n-3)!!}{n!}$$

Note: $(2n-3)!!\,(2n-2)!! = (2n-2)!$

$$= 2^n \frac{(2n-2)!}{n!\,(2n-2)!!} = 2\frac{(2n-2)!}{n!\,(n-1)!} = \frac{2}{n}\binom{2n-2}{n-1}$$

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{\sum_{n\geq 1} \frac{2}{n}\binom{2n-2}{n-1} x^n}{2x} = \sum_{n\geq 1} \frac{1}{n}\binom{2n-2}{n-1} x^{n-1}$$

$m = n-1$

$$= \sum_{m\geq 0} \frac{1}{m+1}\binom{2m}{m} x^m$$

Thm. $C_n = \frac{1}{n+1}\binom{2n}{n}$ for all $n \geq 0$.

The values for $n=0,1,\ldots 9$ are $1, 1, 2, \boxed{5}, 14, 42, 132, 429, 1430, 4862$

Other Catalan objects:

① ways to apply binary operation $*$ to $n+1$ elements:

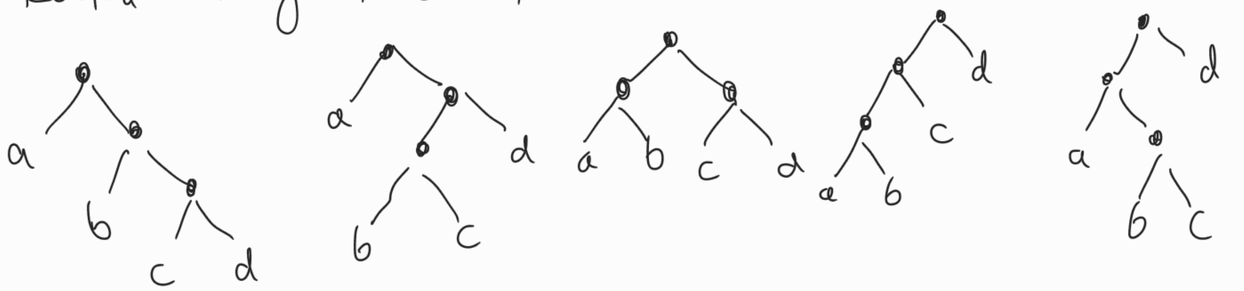
$$a*(b*(c*d)) \quad a*((b*c)*d) \quad (a*b)*(c*d)$$

$$((a*b)*c)*d \quad (a*(b*c))*d$$

Note: This satisfies Catalan recursion:

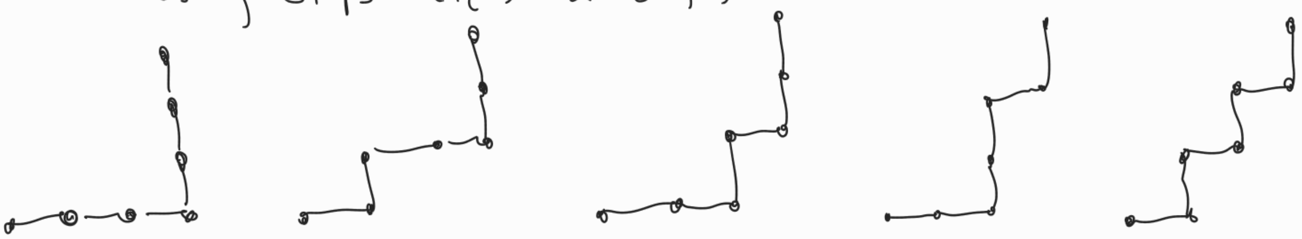
Take last use of $*$, consider left input and right input

② Rooted binary trees w/ $n+1$ leaves




③ Paths from $(0,0)$ to $(n,n)$ which

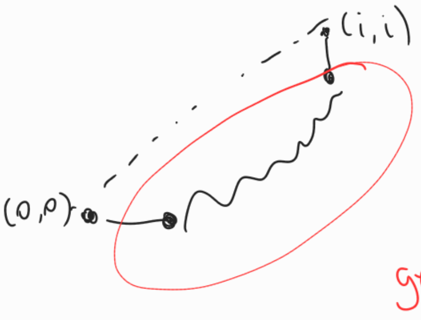
- never go above $x=y$ line
- using steps $(1,0)$ & $(0,1)$



Consider <u>first</u> time path hits $x=y$ line. Call this point $(i,i)$.

Path from $(0,0) \to (i,i)$ is Catalan (not an arbitrary path) does not touch $x=y$ line except at end

Path from $(i,i) \to (n,n)$ is Catalan




$(1,0) \to (i,i-1)$

shift it to left by 1:

get Catalan path $(0,0) \to (i-1,i-1)$