# Rational Generating Functions

**Def.** A formal power series $F(x)$ is *rational* if $\exists$ polynomials $P(x)$, $Q(x)$ s.t. $Q(x) \neq 0$ & $F(x)Q(x) = P(x)$. $\left[F(x) = \frac{P(x)}{Q(x)}\right]$

The *degree* of $F(x)$ is $\deg F = \deg P - \deg Q$.

**Thm.** $F(x) = \sum_{n \geq 0} a_n x^n$. Pick integer $N \geq 0$.

Let $Q(x) = 1 + c_1 x + \cdots + c_r x^r = (1-\gamma_1 x)^{m_1} \cdots (1-\gamma_s x)^{m_s}$

where $\gamma_1, \ldots \gamma_s$ are distinct and nonzero.

The following are equivalent:

(a) For all $n \geq N$, we have $a_{n+r} + \cdots + c_{r-1} a_{n+1} + c_r a_n = 0$

(b) $Q(x)F(x)$ is a polynomial of degree $< N+r$

(c) $\exists$ polynomials $f_1, \ldots f_s$ w/ $\deg f_i < m_i$ s.t. $a_n = \sum_{i=1}^{s} f_i(n) \gamma_i^n$ for all $n \geq N$.

**Pf.** Pick $n \geq N$. The coefficient of $x^{n+r}$ in $Q(x)F(x)$ is

$$a_{n+r} + \cdots + c_{r-1} a_{n+1} + c_r a_n$$

Hence, all of these are $0 \iff Q(x)F(x)$ is a polynomial of degree $< N+r$.

$$(a) \iff (b).$$

Assume (b) holds. Write $F(x) = \frac{P(x)}{Q(x)}$ where $\deg P < N+r$.

Long division $\Rightarrow$ $\exists$ polynomials $g(x)$ & $P_0(x)$ s.t. $\deg g(x) < N$ & $\deg P_0(x) < r$ s.t. $F(x) = g(x) + \frac{P_0(x)}{Q(x)}$

Partial fraction decomposition $\Rightarrow$ $\frac{P_0(x)}{Q(x)} = \sum_{i=1}^{s} \frac{p_i(x)}{(1-\gamma_i x)^{m_i}}$

for some polynomials $p_1(x), \ldots p_s(x)$ s.t. $\deg p_i(x) < m_i$.

If $d < m$, then binomial thm tells us:

$$\frac{x^d}{(1-\gamma x)^m} = \sum_{n\geq 0} \binom{m+n-1}{n} \gamma^n x^{n+d} = \gamma^{-d} \sum_{\boxed{n \geq d}} \binom{m+n-d-1}{n-d} \gamma^n x^n$$

can change to $n \geq 0$

Note: as a polynomial in $n$,

$$\binom{m+n-d-1}{n-d} = \frac{1}{(m-1)!}(m+n-d-1)\cdots(n-d+2)(n-d+1)$$

has roots at $n=0,\ldots,d-1$

$\Rightarrow$ $[x^n]\frac{x^d}{(1-\gamma x)^m} = f(n)\gamma^n$ where $f(n)$ is a polynomial of degree $< m$.

$\Rightarrow$ $[x^n]\frac{P_0(x)}{Q(x)} = \sum_{i=1}^{s} f_i(n)\gamma_i^n$ where $f_i(n)$ is polynomial of degree $< m_i$.

$\Rightarrow$ (c) holds

(c) $\Rightarrow$ (b) by reversing this proof $\square$

When $s=1$ and $\gamma_1 = 1$, (c) tells us $a_n =$ polynomial for $n \geq N$.

Cor. $F(x) = \sum_{n\geq 0} a_n x^n$, pick $N \geq 0$. The following are equivalent.

(a) For all $n \geq N$, we have $\sum_{i=0}^{r} (-1)^{r-i}\binom{r}{i} a_{n+i} = 0$

(b) $(1-x)^r F(x)$ is a polynomial of degree $< N+r$.

(c) $\exists$ polynomial $f$ w/ $\deg f < r$ s.t. $a_n = f(n)$ for all $n \geq N$.

Pf. $Q(x) = (1-x)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} x^i$. $\square$

Ex. $a = (a_1, a_2, a_3) \in \mathbb{Z}^3_{\geq 0}$

$b = (b_1, b_2, b_3) \in \mathbb{Z}^3_{\geq 0}$

$\boxed{a \leq b \text{ means } a_i \leq b_i \text{ for } i = 1,2,3}$

$|b| = b_1 + b_2 + b_3$

Define $F_a(x) = \sum_{\substack{b \text{ s.t.} \\ b \geq a}} x^{|b|} = \frac{x^{|a|}}{(1-x)^3}$

$(b_1 - a_1, b_2 - a_2, b_3 - a_3) \leftrightarrow$ multiset of $[3]$ where we choose $i$ $b_i - a_i$ times

$\Rightarrow \quad \left|\{ b \in \mathbb{Z}^3_{\geq 0} \mid b \geq a \text{ \& } |b| = n \}\right|$ is a polynomial of degree $\leq 2$

for $n \geq |a| - 2$.

$a' = (a_1', a_2', a_3') \in \mathbb{Z}^3_{\geq 0}$. Set $a'' = (\max(a_1, a_1'), \max(a_2, a_2'), \max(a_3, a_3'))$

$$\sum_{\substack{b \text{ s.t.} \\ b \geq a \text{ or } b \geq a'}} x^{|b|} = F_a(x) + F_{a'}(x) - F_{a''}(x) = \frac{x^{|a|} + x^{|a'|} - x^{|a''|}}{(1-x)^3}$$

$\Rightarrow \quad \left|\left\{ b \in \mathbb{Z}^3_{\geq 0} \;\middle|\; \begin{array}{c} b \geq a \text{ or } b \geq a' \\ \text{\& } |b| = n \end{array} \right\}\right|$ is a polynomial in $n$

of degree $\leq 2$ for $n \geq |a''| - 2$.