Binomial Theorem

Goal: $(1+x)^m$, m rational

Lemma. $A(x)$ formal power series s.t. $A(0)=1$, d = positive integer. There exists unique formal power series $B(x)$ s.t. $B(0)=1$ & $B(x)^d = A(x)$.

Notation: $B(x) = A(x)^{1/d}$

pf. Try to solve for $b_1, b_2, \ldots$ in equations coming from comparing coeff. of $B(x)^d = A(x)$. We know that $[x^n](B(x)^d)$ has the form

$$d b_n + f_{n,d} \quad \text{where } f_{n,d} \text{ is a polynomial using } b_1, \ldots, b_{n-1}.$$

Ex.
$$[x^1](B(x)^d) = [x^1] B(x) B(x) \cdots B(x)$$
$$= [x^1](1+b_1x+\ldots)(1+b_1x+\ldots)\cdots(1+b_1x+\ldots)$$
$$= d b_1 \quad (f_{1,d}=0)$$

$$[x^2](B(x)^d) = [x^2](1+b_1x+b_2x^2)(1+b_1x+b_2x^2)\cdots(1+b_1x+b_2x^2)$$
$$= d b_2 + \frac{d(d-1)}{2} b_1 \quad \left(f_{2,d} = \frac{d(d-1)}{2} b_1\right)$$

Have eqns $d b_n + f_{n,d} = a_n$ for $n \geq 1$.

Solve iteratively: $b_n = \dfrac{a_n - f_{n,d}}{d}$ ← uses only $b_1, \ldots b_{n-1}$

base case $(n=1)$: $b_1 = a_1/d$. □

How to define $A(x)^{c/d}$ where c, d integers, $A(0)=1$.

Two ways: ① by Lemma, $A(x)^{1/d}$ exists, so define $(A(x)^{1/d})^c$

② $A(x)^c$ has constant term 1, lemma gives $(A(x)^c)^{1/d}$.

They agree: $((A(x)^{1/d})^c)^d = ((A(x)^{1/d})^d)^c = A(x)^c$

$\Rightarrow (A(x)^{1/d})^c$ is a dth root of $A(x)^c$

It has constant term 1, but dth roots are unique, so

$(A(x)^{1/d})^c = (A(x)^c)^{1/d}$. Define $A(x)^{c/d}$ to be either one.

A similar argument shows $A(x)^{c/d} = A(x)^{ec/ed}$ for any integer e.

Def. m rational, $\binom{m}{0} = 1$, $\binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!}$ $k>0$ integer.

$k! = k(k-1)\cdots 1$

If m = positive integer, $\binom{m}{k} = \frac{m!}{k!(m-k)!}$

$\boxed{\begin{array}{l}\text{Thm (Binomial theorem), } m \text{ rational.} \\ (1+x)^m = \sum_{n\geq 0} \binom{m}{n} x^n\end{array}}$

Note: can compose w/ any formal power series whose constant term is 0

Ex. If $A(0)=1$: $A(x)^m = (1+(A(x)-1))^m = \sum_{n\geq 0} \binom{m}{n}(A(x)-1)^n$

Ex ① m positive integer $(1+x)^m = \sum_{n\geq 0} \binom{m}{n} x^n = 1 + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m$

Note: $\binom{m}{n} = 0$ if $n>m$ b/c in numerator $m(m-1)\cdots(m-n+1)$, get 0

② $m=-1$. $(1+x)^{-1} = \frac{1}{1+x} = \sum_{n\geq 0} \binom{-1}{n} x^n = \sum_{n\geq 0} (-1)^n x^n$

$$\binom{-1}{n} = \frac{(-1)(-2)\cdots(-1-n+1)}{n!} = \frac{(-1)^n\, 1\cdot 2 \cdot \cdots \cdot n}{n!} = (-1)^n$$

Alt: from geom. series, have $\frac{1}{1-x} = \sum_{n\geq 0} x^n$, compose w/ $-x$.

③ $m=-d$, d positive integer.

$$(1+x)^{-d} = \sum_{n\geq 0} \binom{-d}{n} x^n = \sum_{n\geq 0} (-1)^n \binom{d+n-1}{n} x^n$$

$$\binom{-d}{n} = \frac{(-d)(-d-1)\cdots(-d-n+1)}{n!} = (-1)^n \frac{d(d+1)\cdots(d+n-1)}{n!} = \frac{(d+n-1)!}{n!(d-1)!}(-1)^n$$

$$= (-1)^n \binom{d+n-1}{n}$$

$$\Rightarrow \frac{1}{(1-x)^d} = \sum_{n\geq 0} \binom{d+n-1}{n} x^n$$

Ex. $m = 1/2$. $\binom{1/2}{n} = \dfrac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2}-n+1)}{n!}$

$= \dfrac{(-1)^{n-1}\,\frac{1}{2}(\frac{1}{2})(\frac{3}{2})\cdots(n-\frac{3}{2})}{n!} = \dfrac{(-1)^{n-1}\,3\cdot 5\cdots(2n-3)}{2^n n!}$

Notation: $k!! = k(k-2)(k-4)\cdots$ $\quad \dfrac{(-1)^{n-1}(2n-3)!!}{2^n n!}$ $\quad n \geq 2$

$\binom{1/2}{0} = 1$, $\binom{1/2}{1} = 1/2$

$$(1+x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n} x^n = 1 + \frac{1}{2}x + \sum_{n\geq 2} \frac{(-1)^{n-1}(2n-3)!!}{2^n n!} x^n$$

Lemma. m rational, $A(0)=1$, then $D(A(x)^m) = m(DA)\,A(x)^{m-1}$

Pf. $m = p/q$ $\quad p,q$ integers. Then

$$p(DA)A(x)^{p-1} = D(A(x)^p) = D((A(x)^m)^q) = q(A(x)^m)^{q-1} D(A(x)^m)$$

$$\Longrightarrow D(A(x)^m) = \frac{p(DA)\,A(x)^{p-1}}{q(A(x)^m)^{q-1}} = m(DA)\,A(x)^{m-1}. \quad \square$$

$\boxed{\text{PF of BINOMIAL THM}}$ $\quad D((1+x)^m) = m(1+x)^{m-1}$ | $D^n((1+x)^m) = m(m-1)\cdots(m-n+1)(1+x)^{m-n}$

$$[x^n](1+x)^m = \frac{(D^n(1+x)^m)(0)}{n!} = \frac{m(m-1)\cdots(m-n+1)}{n!} = \binom{m}{n} \quad \square$$

Consider eqn $\quad A(x)t^2 + B(x)t + C(x) = 0$

Facts: ① There are at most 2 solutions for t

② If solution exists, it must be given by quadratic formula:

$$t = \frac{-B(x) \pm (B(x)^2 - 4A(x)C(x))^{1/2}}{2A(x)}$$

$\leftarrow$ Is this defined?

$\leftarrow$ Is $A(x)$ invertible?

If $A(x)$ not invertible, this means $A(x)$ divides numerator for some choice of sign.