Redfield – Pólya Theory

G = group, X = set w/ G-action, Y = set of labels

$\Rightarrow$ G acts on $Y^X = \{\text{functions } X \to Y\}$

weight of $f: X \to Y$ is $wt(f) = \prod_{x \in X} f(x)$

Ex. If 2 things blue, 3 are red, $wt = B^2R^3$

If f, f' in same orbit, then $wt(f) = wt(f')$

Define $wt(G \cdot f) = wt(f)$

Let $n = |X|$, given $g \in G$, let $c_{X,i}(g)$ = # cycles of φ_g w/ length i

Note: $c_X(g) = \sum_{i=1}^{n} c_{X,i}(g)$

Let $t_1, \dots, t_n$ be new variables.

cycle indicator of G acting on X is

$$Z_X(G; t_1, \dots t_n) = \frac{1}{|G|} \sum_{g \in G} t_1^{c_{X,1}(g)} t_2^{c_{X,2}(g)} \cdots t_n^{c_{X,n}(g)}$$

Thm. (Redfield – Pólya).

$$\sum_{\mathcal{O}} wt(\mathcal{O}) = Z_X\Big(G; \sum_{y \in Y} y, \sum_{y \in Y} y^2, \dots, \sum_{y \in Y} t^n\Big)$$

[we substitute sum of dth powers of elements of Y in for t_d]

sum over all orbits $\mathcal{O}$ of G acting on X.

Note: if we substitute $y \to 1$ for all $y \in Y$,

LHS = # orbits, RHS = $\frac{1}{|G|} \sum_{g \in G} |Y|^{c_X(g)}$

Pf. Define $S = \{(g,f) \in G \times Y^X \mid g \cdot f = f\}$

Set $wt(S) = \sum_{(g,f) \in S} wt(f)$.

Consider compute $wt(S)$ by summing over Y^X:

$$wt(S) = \sum_{f \in Y^X} |G_f| \, wt(f).$$ Divide by $|G|$:

$$\frac{wt(S)}{|G|} = \sum_{f \in Y^X} \frac{|G_f|}{|G|} wt(f) = \sum_{f \in Y^X} \frac{1}{|G \cdot f|} wt(G \cdot f) = \sum_{\mathcal{O}} wt(\mathcal{O})$$

Compute $wt(S)$ by summing over G:

$$wt(S) = \sum_{g \in G} \sum_{f \in (Y^X)^g} wt(f).$$

$f \in (Y^X)^g$ is constant on cycles of g, call cycles $C_1, \ldots C_r$.

To specify f, we need to pick values $f(C_i)$ for $i = 1, \ldots, r$.

$$wt(f) = f(C_1)^{|C_1|} f(C_2)^{|C_2|} \cdots f(C_r)^{|C_r|}.$$

$$\Rightarrow \sum_{f \in (Y^X)^g} wt(f) = \prod_{i=1}^{r} \left(\sum_{y \in Y} y^{|C_i|} \right) = \prod_{j=1}^{n} \left(\sum_{y \in Y} y^j \right)^{c_{X,j}(g)}$$

Compare to definition of cycle indicator:

$$\frac{wt(S)}{|G|} = \frac{1}{|G|} \sum_{g \in G} \sum_{f \in (Y^X)^g} wt(f) = Z_X\left(G; \sum_{y \in Y} y, \sum_{y \in Y} y^2, \ldots, \sum_{y \in Y} y^n\right) \square$$

Ex. necklaces of length 4. $X = \mathbb{Z}/4$, $G = \mathbb{Z}/4$

$Y = \{y_1, \dots, y_k\}$, $G = \{(0123), (02)(13), (0321), (0)(1)(2)(3)\}$

$$Z_X(G; t_1, t_2, t_3, t_4) = \frac{1}{4}(2t_4 + t_2^2 + t_1^4)$$

$$t_d \to \sum_{i=1}^{k} y_i^d \rightsquigarrow \frac{1}{4}\Big(2\sum_i y_i^4 + \big(\sum_i y_i^2\big)^2 + \big(\sum_i y_i\big)^4\Big)$$

coeff of:

y_1^4	$\frac{1}{2}$	+	$\frac{1}{4}$	+	$\frac{1}{4}$	=	1
$y_1^3 y_2$	0	+	0	+	1	=	1
$y_1^2 y_2^2$	0	+	$\frac{1}{2}$	+	$\frac{3}{2}$	=	2
$y_1^2 y_2 y_3$	0		0		3	=	3
$y_1 y_2 y_3 y_4$	0		0		6	=	6

$y_1^2 y_2^2$: $\begin{smallmatrix} & a & \\ b & & a \\ & b & \end{smallmatrix}$ $\quad$ $\begin{smallmatrix} & a & \\ b & & b \\ & a & \end{smallmatrix}$

$y_1^2 y_2 y_3$: $\begin{smallmatrix} & a & \\ c & & a \\ & b & \end{smallmatrix}$ $\quad$ $\begin{smallmatrix} & a & \\ b & & a \\ & c & \end{smallmatrix}$ $\quad$ $\begin{smallmatrix} & a & \\ c & & b \\ & a & \end{smallmatrix}$

$y_1 y_2 y_3 y_4$: $\begin{smallmatrix} & a & \\ d & & b \\ & c & \end{smallmatrix}$ $\quad$ $\begin{smallmatrix} & a & \\ c & & b \\ & d & \end{smallmatrix}$ $\quad$ $\begin{smallmatrix} & a & \\ d & & c \\ & b & \end{smallmatrix}$ $\quad$ $\begin{smallmatrix} & a & \\ b & & c \\ & d & \end{smallmatrix}$ $\quad$ $\begin{smallmatrix} & a & \\ c & & d \\ & b & \end{smallmatrix}$ $\quad$ $\begin{smallmatrix} & a & \\ b & & d \\ & c & \end{smallmatrix}$

Note: $Z_X(G; \sum y, \dots \sum y^n)$ is symmetric in $y_1, \dots, y_k$

necklaces w/ 1 color = choose color $\cdot$ coeff of $y_1^4 = k$

necklaces w/ 2 colors, one appearing exactly 3 times $=$ choose colors $\cdot$ coeff of $y_1^3 y_2 = k(k-1)$

necklaces w/ 2 colors, both appearing twice $= \binom{k}{2} \cdot 2 = k(k-1)$

necklaces w/ 3 colors one appearing twice $= k\binom{k-1}{2} \cdot 3$

necklaces w/ 4 colors $= \binom{k}{4} 6$

Ex. necklaces of length 4 up to reflection

$X = \mathbb{Z}/4, \quad G = D_4$

new elements $\quad G \setminus \mathbb{Z}/4 = \{(0)(13)(2), (0)(12)(3), (01)(23), (03)(12)\}$

$$Z_X(D_4; t_1, t_2, t_3, t_4) = \frac{1}{8}\left(2t_4 + 3t_2^2 + t_1^4 + 2t_1^2 t_2\right)$$

sub: $t_d \to \sum_{i=1}^{k} y_i^d \rightsquigarrow \frac{1}{8}\left(2\sum y_i^4 + 3\left(\sum y_i^2\right)^2 + \left(\sum y_i\right)^4 + 2\left(\sum y_i\right)^2\left(\sum y_i^2\right)\right)$

coeff of:

y_1^4	$\frac{1}{4}$ +	$\frac{3}{8}$ +	$\frac{1}{8}$ +	$\frac{1}{4}$	$= 1$
$y_1^3 y_2$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$= 1$
$y_1^2 y_2^2$	0	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$= 2$
$y_1^2 y_2 y_3$	0	0	$\frac{3}{2}$	$\frac{1}{2}$	$= 2$
$y_1 y_2 y_3 y_4$	0	0	3	0	$= 3$