Group actions

group: set G w/ binary operation (associative, has unit, inverses)

subgroup: nonempty subset $H \subseteq G$ closed under binary operation and taking inverses

Ex. X = finite set, $\mathfrak{S}_X = \{f : X \to X \mid f \text{ bijective}\}$ w/ composition as binary operation.

If $X = [n]$, $\mathfrak{S}_n = \mathfrak{S}_{[n]}$ (symmetric group)

Def. G group, X = set. An <u>action</u> of G on X is a function $\varphi : G \times X \to X$ (written as $g \cdot x$ instead of $\varphi(g, x)$)

s.t. (1) $1 \cdot x = x \quad \forall x \in X$

(2) $g \cdot (h \cdot x) = (gh) \cdot x \quad \forall x \in X,\ g, h \in G.$

$\Rightarrow$ each $g \in G$ gives bijection $\varphi_g : X \to X$ by $\varphi_g(x) = g \cdot x$

Prop. The assignment $g \mapsto \varphi_g$ is a group homomorphism $G \to \mathfrak{S}_X$. Conversely, every group homomorphism $G \to \mathfrak{S}_X$ is of this form.

Pf. First, φ_1 is identity function on X by (1).

Second, for any $x \in X$, $g, h \in G$, have

$$\varphi_g(\varphi_h(x)) = g \cdot (h \cdot x) \overset{(2)}{=} (gh) \cdot x = \varphi_{gh}(x)$$

$$\Rightarrow \varphi_g \varphi_h = \varphi_{gh} \Rightarrow \varphi_g \varphi_{g^{-1}} = \varphi_1 \Rightarrow \varphi_g^{-1} = \varphi_{g^{-1}}.$$

Conversely, suppose $\varphi : G \to \mathfrak{S}_X$ is a homomorphism.

For $g \in G$ and $x \in X$, define $g \cdot x = \varphi(g)(x)$

Since $\psi(1) = id_X$, have $1 \cdot x = x \ \forall x \in X \Rightarrow (1)$

Second, for $g, h \in G$ $\psi(gh) = \psi(g)\psi(h)$, so for $x \in X$, have $(gh)\cdot x = \psi(gh)(x) = \psi(g)(\psi(h)(x)) = g\cdot(h \cdot x) \Rightarrow (2)$. $\square$

Given action of G on X, each g is a permutation, so can define #cycles of g (this depends on action)
$$\#\text{cycles of } g = c_X(g)$$

Given $x \in X$, the <u>orbit</u> of x is
$$G \cdot x = \{ g \cdot x \mid g \in G \}$$

Being in same orbit is an equivalence relation

The <u>stabilizer</u> of $x \in X$ is
$$G_x = \{ g \in G \mid g \cdot x = x \}$$

G_x is a subgroup of G.

<u>Lemma</u>. (orbit-stabilizer formula). $\dfrac{|G|}{|G_x|} = |G \cdot x|$

<u>Pf</u>. Let $\{x_1, \dots, x_r\} = G \cdot x$. For each $i = 1, \dots, r$, $\exists g_i \in G$ s.t. $g_i \cdot x = x_i$. For any $g \in G$, $g \cdot x = g_i \cdot x$ for some i, then $g_i^{-1} g \cdot x = x \Rightarrow g_i^{-1} g \in G_x$. In particular, every element of G is uniquely of the form $g_i h$ for some $h \in G_x$ and some $1 \le i \le r$: if $g_i h = g_j h' \Rightarrow$
$$g_j^{-1} g_i = h' h^{-1} \in G_x \Rightarrow g_j^{-1} g_i x = x \Rightarrow g_i x = g_j x$$
$$\Rightarrow i = j \Rightarrow h = h'!$$

$$\Rightarrow |G| = r\,|G_x| \Rightarrow \frac{|G|}{|G_x|} = |G\cdot x|. \qquad \square$$

Ex. $G = S_n$, $X = [n]$, $\sigma \cdot i = \sigma(i)$

Orbit of any element is X.

Stabilizer of i is permutations satisfying $\sigma(i) = i$

$\Rightarrow (n-1)!$ many of them.

orbit-stabilizer: $\frac{n!}{(n-1)!} = n$

Ex. D_4 = dihedral group of order 8

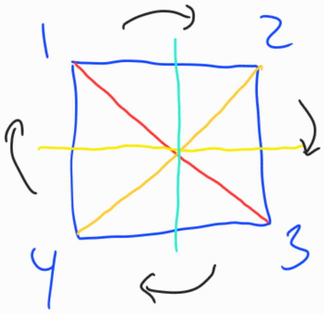
G = symmetries of a square.

X = set of vertices of the square

orbit of any vertex is whole set

$\Rightarrow$ orbit stabilizer: stabilizer of any vertex has size 2.

(identity, reflection across diagonal containing that vertex)



		#cycles [cycle notation]
identity:	(1)(2)(3)(4)	4
rotations:	(1 2 3 4)	1
	(1 3)(2 4)	2
	(1 4 3 2)	1
reflections	(1)(3)(2 4)	3
	(2)(4)(1 3)	3
	(1 4)(2 3)	2
	(1 2)(3 4)	2

Ex. Pick $n \geq 2$. $G = \mathfrak{S}_n$, $X = [n] \times [n] = \left\{ (i,j) \;\middle|\; \begin{matrix} 1 \leq i \leq n \\ 1 \leq j \leq n \end{matrix} \right\}$

$\sigma \cdot (i,j) = (\sigma(i), \sigma(j))$

2 orbits: ① $\{ (i,j) \mid i \neq j \}$ [if $i \neq j$ & $k \neq \ell$, $\exists\, \sigma \in \mathfrak{S}_n$ s.t. $\sigma(i) = k$, $\sigma(j) = \ell$]

② $\{ (i,i) \}$

Stabilizer of (i,j) w/ $i \neq j$ is $\{ \sigma \mid \sigma(i) = i, \sigma(j) = j \}$

$\Rightarrow$ has size $(n-2)!$

orbit-stabilizer: $\dfrac{n!}{(n-2)!} = n(n-1)$ ✓

Variation: $X =$ 2-element subsets of $[n]$

Only one orbit (whole set)

stabilizer of subset $\{i,j\}$ is $\left\{ \sigma \;\middle|\; \begin{matrix} \sigma(i) = i, \sigma(j) = j \\ \underline{\text{OR}} \;\; \sigma(i) = j, \sigma(j) = i \end{matrix} \right\}$

$\Rightarrow$ size is $2(n-2)!$

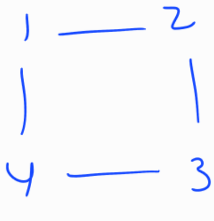
orbit-stabilizer: $\dfrac{n!}{2(n-2)!} = \binom{n}{2}$

Ex. $X =$ vertices of a simple graph Γ.

$G = \{ \sigma \in \mathfrak{S}_X \mid \forall i,j,\ \{i,j\}$ is an edge $\Leftrightarrow \{\sigma(i), \sigma(j)\}$ is an edge $\}$

$\searrow$ subgroup of $\mathfrak{S}_X$ (automorphism group of Γ $\mathrm{Aut}(\Gamma)$)

For example: $X = 1,2,3,4$, $\Gamma =$



$\mathrm{Aut}(\Gamma) = D_4$